

Coherence and interactions in diffusive systems

Gilles Montambaux

Laboratoire de Physique des Solides, Univ. Paris-Sud, CNRS, UMR 8502, F-91405 Orsay Cedex, France. – montambaux@lps.u-psud.fr

Lectures notes for the "International School on Physics of Zero and One Dimensional Nanoscopic Systems", Saha Institute of Nuclear Physics, Calcutta, India, 1-9 February, 2006

This lecture is a tutorial introduction to coherent effects in disordered electronic systems. Avoiding technicalities as most as possible, I present some personal points of view to describe well-known signatures of phase coherence like weak localization correction or universal conductance fluctuations. I show how these physical properties of phase coherent conductors can be simply related to the classical return probability for a diffusive particle. The diffusion equation is then solved in various appropriate geometries and in the presence of a magnetic field. The important notion of quantum crossing is developed, which is at the origin of the quantum effects. The analogy with optics is exploited and the relation between universal conductance fluctuations and speckle fluctuations in optics is explained. The last part concerns the effect of electron-electron interactions. Using the same simple description, I derive qualitatively the expressions of the Altshuler-Aronov anomaly of the density of states, and of the correction to the conductivity. The last part, slightly more technical, addresses the question of the lifetime of a quasiparticle in a disordered metal.

1 Introduction : phase coherence and disorder

Although the topic of this School mainly concerns nanoscopic systems, this set of lectures is devoted to an intermediate range, between the nanoscopic and macroscopic scales, the so-called mesoscopic regime [1]. In this regime, the system to be considered may be large compared to the mean free path of the electrons. Disorder plays then a very important role and, in the so-called diffusive regime, the interplay between disorder and quantum interference effects is crucial. This is the main subject of these lectures. Here, electronic interactions will be treated as a perturbation, in contrast with other topics discussed in this School where the electronic correlations may play the most important role. I will try to present some personal points of view in order to describe these well-known signatures of phase coherence like weak localization

or universal conductance fluctuations. The goal of these lectures is to avoid technicalities as most as possible. The last part concerns the effect of electron-electron interactions.

To describe interference effects in electronics, it is useful to compare with simple facts known in optics. The simplest experiment with light is the two-slit Young experiment and phase coherent effects considered in electronics are nothing but some more sophisticated versions of the Young experiment. This two-slit experiment can be also performed with electrons in vacuum [2] but here we shall consider metallic wires, that are complex disordered media.

In vacuum, an electron beam is split in two parts and the intensity is measured on a screen. The topological equivalent in a metal consists in a loop pierced by a magnetic flux and we measure the current resulting from the interferences between the two paths, see figure 1. In optics the way to probe the interference pattern on the screen is to change the optical path between the two trajectories, by changing the nature of the medium, that is its optical index. For electrons, the charge is coupled to the vector potential \mathbf{A} and these interference pattern can be modified with a magnetic field.

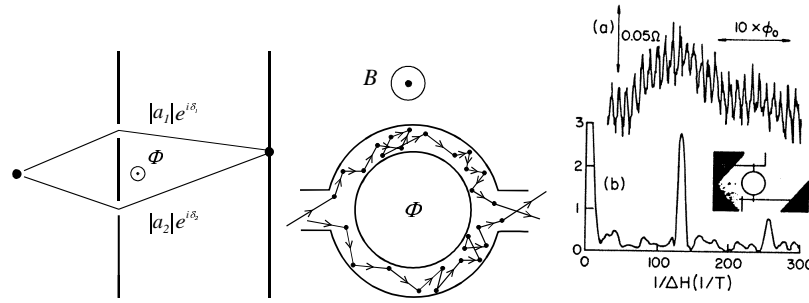


Fig. 1. *Left : schematic representation of the Aharonov-Bohm effect. A flux tube of flux ϕ is placed behind the two slits. Center : Schematic equivalent of the Aharonov-Bohm experiment in a metallic ring. Right : Magnetoresistance oscillations of a Au ring and its Fourier transform [3].*

If we try to transpose what is known from optics to electronics, we measure a current intensity which is proportional to the probability for the electrons to traverse the loop. To calculate this probability in quantum mechanics, we have to add the contributions of two quantum amplitudes corresponding to the two sides of the loop, and the current (a probability) is proportional to the square of this quantum amplitude. Each quantum amplitude ψ_i has a phase φ_i :

$$\psi_1 = \psi e^{i\varphi_1} \quad , \quad \psi_2 = \psi e^{i\varphi_2} \quad . \quad (1)$$

We have to sum the amplitudes and take the modulus square. For the current, we get a classical term plus an interference term :

$$I \propto |\psi_1 + \psi_2|^2 = 2\psi^2 [1 + \cos(\phi_1 - \phi_2)] \quad . \quad (2)$$

Classically, the current should be the sum of the two currents, this corresponds to Ohm's law, but quantum mechanically there is some additional phase effect. And if we apply a magnetic field (this is the well-known Aharonov-Bohm effect [5]), there is an additional phase along each of the two trajectories

$$\delta\varphi_1 = \frac{e}{\hbar} \int_1 \mathbf{A} \cdot d\mathbf{l} \quad , \quad \delta\varphi_2 = \frac{e}{\hbar} \int_2 \mathbf{A} \cdot d\mathbf{l} \quad , \quad (3)$$

so that the phase difference is modulated by the circulation of the vector potential along the closed loop formed by the two trajectories,

$$\Delta\varphi = \delta\varphi_1 - \delta\varphi_2 = \frac{e}{\hbar} \oint \mathbf{A} \cdot d\mathbf{l} = 2\pi \frac{\phi}{\phi_0} \quad , \quad (4)$$

and is proportional to the magnetic flux ϕ through the loop. $\phi_0 = h/e$ is the flux quantum. This tells us that the wave functions, the energy levels, thermodynamic and transport properties must be function of this flux with period ϕ_0 . Of course this does not tell us anything about the amplitude of these oscillations. Unlike the case of the Young experiment with light or for electrons in vacuum, the problem here is much more complicated due to disorder. The question now is : *do interference effect survive in the presence of disorder?* Do oscillations persist?

Indeed, some oscillations remain in the presence of disorder. The pionner experiment founding the field of mesoscopic physics was performed by Webb *et al.* in 1985 [3]. They measured the resistance of a ring of micronic size, in the presence of a magnetic field (figure 1). They found that the resistance oscillates with the field, proving the existence of an interference effect, even in the presence of disorder. Two interesting features must be noted : firstly, the period of the oscillations is the flux quantum ϕ_0 as expected. Secondly, the typical amplitude of the oscillations of the conductance is $\Delta G = \Delta R/R^2 \simeq e^2/h$.

The necessary condition for these oscillations to exist is that phase coherence is preserved, a condition which is obeyed in vacuum. But in a metal, this phase coherence is broken because electrons interact with other degrees of freedom (*e.g.* phonons, other electrons, magnetic impurities). Because of this coupling, each electron can lose the memory of its phase. This happens on a typical length, called the *phase coherence length*, denoted by L_ϕ , which depends on the coupling to these degrees of freedom. Typically at 1K, it is of order of 1 μ . This is the *mesoscopic range*. Of course, such oscillations

do not exist for a macroscopic ring, because phase coherence is lost at the macroscopic level.

There is another important scale, due to static disorder, over which electrons experience collisions over impurities. This scale l_e is named the *elastic mean free path*. It is much smaller than the phase coherence length. Since elastic collisions do not break phase coherence, electrons may experience many collisions without losing the memory of their phase. Each electron keeps the memory of its phase typically on a scale $L_\phi \gg l_e$. The physics we are going to discuss corresponds to length scales which are much larger than the elastic mean free path l_e , but smaller than L_ϕ , so that the sample can be considered as quantum mechanically coherent. This regime is called the *mesoscopic regime*.

We want to construct a theory for these oscillations. How to explain their amplitude? How can they survive disorder? Another important key experiment was performed by Sharvin and Sharvin on a cylinder [4], before the Webb's experiment on a single ring. In this case, there are also oscillations, but instead of being ϕ_0 -periodic, they have a period $\phi_0/2$. The fundamental ϕ_0 disappears and the experiment reveals the second harmonics of these oscillations.

There is a simple way to understand this frequency doubling : assume that the cylinder can be viewed as a pile of independent rings, it realizes an average of the oscillations of several independent rings (similar effect for a network of rings [6]). For a given ring, the oscillations have a given phase which depends on the disorder configuration (This phase is 0 or π for a two-terminal geometry [7]). For another ring, the oscillations have a different phase. When averaging over several rings (as is done in a cylinder), because of this random phase, the oscillations vanish.

So, disorder is destructive for quantum interferences. However this simple argument would tell us that no oscillations should survive disorder averaging. This is not the case since the cylinder experiment shows oscillations with period $\phi_0/2$. In average, there are still oscillations, but with period $h/2e$. This simple and very important fact tells us that *some contributions survive disorder averaging*. So the question is : how is it possible that some robust contribution survive disorder averaging ?

2 Important scales

We shall consider weakly disordered metals, such that the average distance between two collision events is much larger than the Fermi wave length : $l_e \gg \lambda_F$. This condition allows for a semiclassical description of electronic waves. Moreover we assume that the typical size L of the system is much larger than the mean free path l_e so that the electronic motion in the sample is diffusive : electrons collide elastically many times while traversing the system. Finally

Webb <i>et al.</i> experiment	Sharvin-Sharvin experiment
↓	↓
Even in the presence of disorder, phase coherence is preserved on distances much larger than l_e → sample specific interference effects, period ϕ_0 .	Some interference effects survive disorder averaging, period $\phi_0/2$ → pairing of trajectories.

Fig. 2. Conclusions that can be drawn from the two pionnieriing experiments in mesoscopic physics.

we assume that the system is completely phase coherent, that is $L \ll L_\phi$. To summarize, the system is weakly disordered, diffusive and mesoscopic :

$$\lambda_F \ll l_e \ll L \ll L_\phi . \quad (5)$$

We shall see that phase coherent effects may not disappear but are simply reduced in the macroscopic limit $L_\phi < L$, so that we shall also consider the case :

$$\lambda_F \ll l_e \ll L_\phi \ll L . \quad (6)$$

In strong disorder, when the mean free path becomes of order of the Fermi wave length $k_F l_e \simeq 1$, interference effects are strong and lead to localization of the electronic waves. This is the domain where the electronic states are *exponentially localized* in space, with Anderson localization from extended to localized waves. This topic will not be covered here. In the opposite regime of very weak disorder, the mean free path becomes so large (or the system is so small) that the mean free path becomes larger than the system size. Collisions with impurities are rare, and occur mainly on the boundaries of the system. This is the so-called *ballistic regime*, where the physics is mainly driven by the structure of the boundaries, *i.e.* the shape of the system. For most shapes, the trajectories are chaotic. A common method to describe this regime is the so-called Random Matrix Theory also used in other fields of physics like nuclear physics. There are quite interesting common features between some aspects of transport in chaotic dots and nuclear physics. Let us also emphasize that the Random Matrix Theory of scattering or transmission matrices can also be used to describe diffusive systems [8].

Since the typical size L of the system is much larger than the mean free path, the electronic motion is diffusive. The average distance between collision events, the mean free path is related to the collision time τ_e , $l_e = v_F \tau_e$, v_F being the Fermi velocity, since the electronic motion between two collisions

is ballistic. For times much longer than the collision time τ_e , the motion is diffusive and the typical distance an electron can reach after a time t scales like

$$r^2 = Dt \quad , \quad (7)$$

where D is the diffusion coefficient given by $D = v_F l_e / d$, d being the space dimensionality. This relation tells us that for a finite system of size L , a very important scale appears : this is the time for which an electron typically sees the boundaries of the system. It is called the traversal time, or *Thouless time*. It is the time for an electron to "realize" that the system is finite. It is given by

$$\tau_D = \frac{L^2}{D} \quad . \quad (8)$$

To this characteristic time, is associated a characteristic energy, the *Thouless energy* E_c :

$$E_c = \frac{\hbar}{\tau_D} = \frac{\hbar D}{L^2} \quad . \quad (9)$$

This energy scale plays a major role in the description of thermodynamic and transport properties of mesoscopic diffusive systems. For time scales smaller than τ_D , the electron propagates like in infinite space. The diffusive motion depends on the space dimensionality of the system. On the other hand, in the long time scale, the electronic motion explores the entire system, this is the so-called ergodic regime.

Using eq.(7), we can associate to the phase coherence length L_ϕ a characteristic time, the phase coherence time τ_ϕ :

$$\tau_\phi = \frac{L_\phi^2}{D} \quad . \quad (10)$$

This is the time during which an electron keeps the memory of its phase.



Fig. 3. Characteristic energy scales defining the different regimes studied in coherent multiple scattering. Explanations are given in the text.

Figure 3 presents a scale of characteristic energies (or inverse characteristic times). At short time scales, the motion is ballistic. For times larger than τ_e , the motion is diffusive in free space. Above τ_D , the motion is bounded, this is the ergodic regime. Then τ_ϕ separates the mesoscopic regime and the classical regime. If $\tau_\phi > \tau_D$ ($L_\phi > L$), the system is mesoscopic.

In the diffusive regime, the space dimensionality d plays an important role. Moreover, the one-dimensional case is somehow special. In strictly one dimension, it is known that there is no diffusive regime since, in the presence of disorder, all states are exponentially localized. So we shall not consider this case but rather *quasi*-one-dimensional systems, with a transverse width a , so that the real motion is three-dimensional (the proper Schrödinger equation to be solved would be three-dimensional with a quantization of the transverse component of the wave vector), but the diffusion is one-dimensional. Instead of having one transverse propagation channel, there are many transverse channels (the transverse size is much larger than λ_F). At short time (smaller than a "transverse" Thouless time a^2/D), diffusion is three-dimensional, but at larger times, there is a one-dimensional propagation of the diffusion cloud. When considering the transport through a wire ($L \gg a$), we shall be interested in time scales necessary to traverse the wire, that is times of order of τ_D , which is much larger than the transverse time, so that we can consider that at this time scale the diffusion is one-dimensional.

3 Classical probability and diffusion equation

The aim of these lectures is to propose a qualitative description of physical phenomena, trying to avoid sophisticated tools and keeping in mind that we are essentially concerned by the calculation of disordered averaged quantities.

An average quantity like the conductance basically measures the probability for electrons to cross the system. What is the nature of this probability? Let us first spend some time to describe the probability $P(\mathbf{r}, \mathbf{r}')$ which describes the propagation of a particle from a point \mathbf{r} to a point \mathbf{r}' . In quantum mechanics, this propagation is described by a probability *amplitude*. This amplitude is called a Green's function $G(\mathbf{r}, \mathbf{r}')$. We do not aim to develop the theory of Green's functions. For our purpose here, it is sufficient to note that there are many possible scattering trajectories from \mathbf{r} to \mathbf{r}' . Thus a Green's function has the following structure : it is the sum of all the probability amplitudes corresponding to various multiple scattering trajectories from \mathbf{r} to \mathbf{r}' , each trajectory being characterized by an amplitude and a phase proportional to its action, that is its length [9] :

$$G(\mathbf{r}, \mathbf{r}') = \sum_j A_j(\mathbf{r}, \mathbf{r}') \quad . \quad (11)$$

Now, we want to know the probability to find a particle at point \mathbf{r}' if it has been injected at point \mathbf{r} . The probability to go from \mathbf{r} to \mathbf{r}' is the modulus square of the amplitude. From eq. (11), we see that this probability is the sum of amplitude squared terms, plus interference terms which pair different trajectories j and j' :

$$|G(\mathbf{r}, \mathbf{r}')|^2 = \sum_{j,j'} A_j(\mathbf{r}, \mathbf{r}') A_{j'}(\mathbf{r}, \mathbf{r}') = \sum_j |A_j(\mathbf{r}, \mathbf{r}')|^2 + \sum_{j' \neq j} A_j(\mathbf{r}, \mathbf{r}') A_{j'}^*(\mathbf{r}, \mathbf{r}') \quad (12)$$

which is an obvious generalization of eq. (2) for the two-slit configuration. Since we know that, in quantum mechanics, one must add amplitudes instead of intensities, the interference term (the second term in eq. 12) cannot be *a priori* neglected. This second term describes interferences between *different* trajectories j and j' . Each contribution in this sum has a random phase which depends on the detail of the impurity configuration. Since the phases are uncorrelated, at the first level of approximation, we may expect that the contribution of the interference term cancels upon disorder averaging. So quantum effects seem not to be so important because of the vanishing of this contribution. We shall see however that this is not exactly the case. Within this approximation, the second term cancels and the probability is essentially given by the sum of intensities :

$$\overline{|G(\mathbf{r}, \mathbf{r}')|^2} = \sum_j \overline{|A_j(\mathbf{r}, \mathbf{r}')|^2} . \quad (13)$$

We see that the phases have disappeared. So the remaining term is completely classical. Indeed, let us assume that some event changes the phase of the amplitude A_j . The complex amplitude A_j^* gets the opposite phase, leaving the probability unchanged.

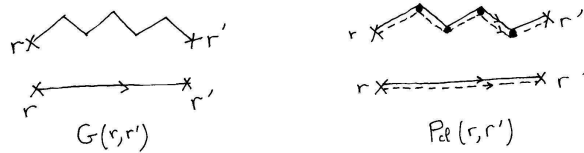


Fig. 4. Schematic representations of a Green's function $G(\mathbf{r}, \mathbf{r}')$ and of the classical probability $P_{cl}(\mathbf{r}, \mathbf{r}') \propto \sum_j |A_j(\mathbf{r}, \mathbf{r}')|^2$. The upper diagrams exhibit a few collision events, which are not represented on the lower diagrams.

To have a simple picture of eq. [13], we represent on figure 4 a quantum amplitude as a line (it is rather a sort of Brownian trajectory). Its complex conjugate is represented as a dashed line. The first term in eq. (13) corresponds to the pairing of a trajectory with its complex conjugate, and we see immediately why the phase disappears. The quantity $\sum_j |A_j(\mathbf{r}, \mathbf{r}')|^2$ resembles the classical probability. We call it a "Diffuson". To be more precise, but without any proof, we define the probability $P(\mathbf{r}, \mathbf{r}', \omega)$ as

$$P(\mathbf{r}, \mathbf{r}', \omega) = \frac{1}{2\pi\rho_0} \overline{G_\epsilon(\mathbf{r}, \mathbf{r}') G_{\epsilon-\omega}^*(\mathbf{r}', \mathbf{r})} \quad (14)$$

The Green's function and its complex conjugate are taken at different energies (or frequencies) ϵ and $\epsilon - \omega$. One can check that this probability is correctly normalized, that is $\int_0^\infty P(\mathbf{r}, \mathbf{r}', t) d\mathbf{r}' = 1$, where $P(\mathbf{r}, \mathbf{r}', t)$ is the Fourier transform of $P(\mathbf{r}, \mathbf{r}', \omega)$. Starting from the Schrödinger equation in a random potential and after disorder averaging, it is possible to show that in the limit $k_F l_e \gg 1$, the probability $P(\mathbf{r}, \mathbf{r}', \omega)$ defined by (14) reduces to the Diffuson $P_{cl}(\mathbf{r}, \mathbf{r}') \propto \sum_j |A_j(\mathbf{r}, \mathbf{r}')|^2$. For slow spatial variations, $P_{cl}(\mathbf{r}, \mathbf{r}', \omega)$ is the solution of a classical diffusion equation :

$$(-i\omega - D\Delta) P_{cl}(\mathbf{r}, \mathbf{r}', \omega) = \delta(\mathbf{r} - \mathbf{r}') \quad (15)$$

where D is the diffusion coefficient. Doing this, we have only considered classical contributions to the average (14). We shall study later the corrections to this classical probability.

Among the solutions of this diffusion equation, one very important is the return probability which enters in many physical quantities. It is the probability $P(\mathbf{r}, \mathbf{r}, t)$ for an electron to return to its original position after time t . I will also consider the space integrated return probability :

$$P(t) = \int P(\mathbf{r}, \mathbf{r}, t) d\mathbf{r} \quad (16)$$

In free space, the solutions of equation (15) are simply obtained from the Fourier transform

$$\left(\frac{\partial}{\partial t} + Dq^2 \right) P(\mathbf{q}, t) = \delta(t) \quad (17)$$

whose solution $P(\mathbf{q}, t)$ is simply

$$P(\mathbf{q}, t) = e^{-Dq^2 t} \quad (18)$$

Fourier transforming back, we find easily

$$P(\mathbf{r}, \mathbf{r}', t) = \frac{1}{(4\pi Dt)^{d/2}} e^{-|\mathbf{r} - \mathbf{r}'|^2 / 4Dt} \quad (19)$$

so that the return probability is given by

$$P(\mathbf{r}, \mathbf{r}, t) = \frac{1}{(4\pi Dt)^{d/2}} \quad \text{and} \quad P(t) = \frac{\Omega}{(4\pi Dt)^{d/2}} \quad (20)$$

where Ω is the volume of the system. The dependence on the dimensionality d of the return probability is crucial since it will explain why dimensionality plays a so important role in mesoscopic physics of diffusive systems.

4 Conductance

Now I wish to come to very simple and qualitative considerations about the conductance of a disordered system, which will be useful for these lectures.

4.1 Classical conductance as the ratio of two volumes

Consider the conductance G . Since it has the dimensions of e^2/h , we can introduce a dimensionless conductance g as

$$g = G/(se^2/h) \quad . \quad (21)$$

Since this quantity is dimensionless, it may be written as the ratio of two physical quantities. For example, by simple manipulations, it can be written as the ratio of two energies : $g \propto E_c/\Delta$, the Thouless energy E_c and the average level spacing Δ . Here I would like to write it as the *ratio of two volumes*. Let us start with the classical Drude conductivity σ_0 . From Einstein relation, it is given by

$$\sigma_0 = se^2 D \rho_0 \quad , \quad (22)$$

where D is the diffusion coefficient and ρ_0 is the density of states at the Fermi level for one spin direction. The factor $s = 2$ accounts for spin degeneracy. By Ohm's law, the conductance G for a three-dimensional system is given by $G = \sigma_0 S/L$, S being the section and L the length of the sample. More generally for a hypercube of typical size L in d dimensions, it is given by $G = \sigma_0 L^{d-2}$. Introducing the Thouless time τ_D defined by (8), let us rewrite the conductance as

$$G = se^2 \rho_0 L^d / \tau_D \quad . \quad (23)$$

The density of states at the Fermi level ρ_0 can be written as $\rho_0 = dA_d/2\pi\lambda_F^{d-1}\hbar v_F$, where λ_F is the Fermi wavelength, v_F is the Fermi velocity, and A_d is the volume of the unit sphere ($A_3 = 4\pi/3$, $A_2 = \pi$, $A_1 = 2$). An easy way to recover immediately this result is to say that the total number of states is $(k_F L)^d$, so that by derivation with respect to the energy, we have necessarily $\rho_0 \propto k_F^d/\epsilon_F \simeq k_F^{d-1}/\hbar v_F \propto 1/\hbar v_F \lambda_F^{d-1}$. As a result, the dimensionless conductance g can be written as

$$g = dA_d \frac{\Omega}{\lambda_F^{d-1} v_F \tau_D} \quad , \quad (24)$$

where $\Omega = L^d$ is the volume of the system. The dimensionless conductance quantity appears as the *ratio of two volumes*, the volume Ω of the system and the volume of a tube of length $v_F \tau_D$ and of section λ_F^{d-1} . We shall see later that this formulation will be quite useful to measure the importance of interference effects.

4.2 Conductance and transmission

Our starting point to describe electric transport is the Landauer formalism. Even staying at a very qualitative level, this formalism is quite natural since

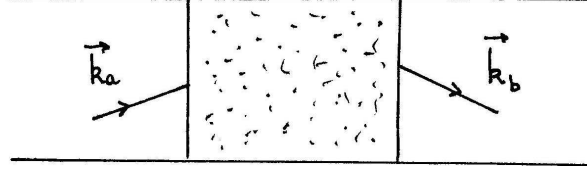


Fig. 5. In the Landauer formalism, the conductance is related to the transmission coefficient between different incoming and outgoing channels.

it expresses the *conductance* as a *transmission coefficient* through the disordered sample.

Consider a disordered conductor of length L and section $S = W^{d-1}$. It is connected to perfect conductors (figure 5) which can be considered as wave guides where free electronic waves propagate. In this geometry, the transverse wave vectors of the eigenmodes (also called channels) are quantized by transverse boundary conditions. One can define a transmission coefficient T_{ab} from an incoming channel a (ingoing wave vector \mathbf{k}_a) to an outgoing channel b (wave vector \mathbf{k}_b). The Landauer formula reads :

$$G = s \frac{e^2}{h} \sum_{a,b} T_{ab} . \quad (25)$$

To calculate the number of transverse channels, one considers that electrons are injected at the Fermi energy, *i.e.* such that $|\mathbf{k}_a| = |\mathbf{k}_b| = k_F$. The transverse component is quantized in units of $2\pi/W$. This quantization imposes the number of channels. In $d = 2$ and $d = 3$, their number is

$$M_2 = \frac{2\pi k_F}{2\pi/W} = k_F W \quad M_3 = \frac{\pi k_F^2}{4\pi^2/W^2} = \frac{k_F^2 S}{4\pi} . \quad (26)$$

Let us consider now the structure of the transmission coefficient T_{ab} . It is the square of an amplitude and it has, with minor differences, the same structure as the probability $P(\mathbf{r}, \mathbf{r}', \omega)$. The main difference is the following : instead of injecting a particle at a point \mathbf{r} inside the sample, we inject a plane wave \mathbf{k}_a from outside the sample. In particular, the boundary conditions have to be treated properly. But, without entering into details, we may easily understand that, after disorder averaging, the average transmission coefficient and consequently the conductance can be related to the probability to cross the sample. More precisely for a $3d$ sample, one can show that the dimensionless conductance is [1]

$$g = \frac{4}{9} M v_F P(0, L) \quad (27)$$

where $P(0, L)$ is the solution of the diffusion equation (15) with appropriate boundary conditions. It is given by $P(0, L) = l_e^2/DL$ so that

$$g = \frac{4}{3} M \frac{l_e}{L} . \quad (28)$$

To obtain these results quantitatively, there are some technicalities that we do not describe here [1]. What should be remembered is the message of fig. 6 : the conductance is proportional to the classical probability to cross the sample. This statement is sufficient to understand how coherence effects appear.

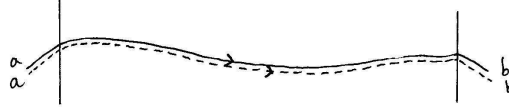


Fig. 6. The conductance is proportional to the classical probability to transmit channel a to channel b (summed over channels). The object which represents this probability is the sum of contributions of paired trajectories as introduced in figure 4. We call it a "Diffuson".

5 Quantum corrections and quantum crossings

The Diffuson is a classical object. It does not depend on the phases of the complex amplitudes. In the diffusive regime, it is solution of a diffusion equation. However, we may have to check whether we have not left aside additional effects when throwing out all the interference terms in relation (12). It turns out that some of these terms have quite interesting consequences.

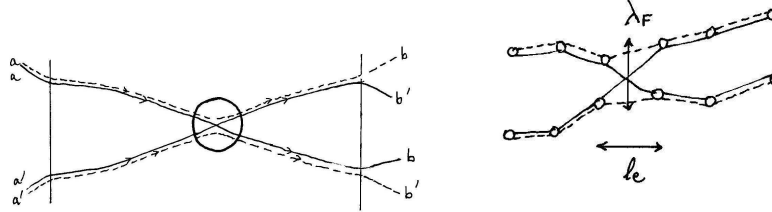


Fig. 7. a) Crossing of two Diffusons. b) Detail : the volume of the intersection region is proportional to $\lambda_F^{d-1} l_e$.

Indeed, quantum effects can appear when two Diffusons cross, or when a Diffuson crosses with itself. The notion of *quantum crossing* is extremely important because it is the source of quantum effects. The Diffuson being a

classical object, coherence effects can only appear because of these quantum crossings. They are at the origin of the weak localization correction and of universal conductance fluctuations. Let us try to get some intuition about these crossing events.

Figure 7.a shows that a crossing mixes four complex amplitudes which belong to two incoming Diffusons and pair them differently. The two emerging Diffusons are built with amplitudes A_j and $A_{j'}$ coming respectively from each of the incoming Diffusons. They have the same phase since they follow the same path. The quantum crossing, also often called a Hikami box in a more technical context, is thus an object whose role is to permute the quantum amplitudes. It is necessarily short ranged, because trajectories have to be as close as possible to each other to avoid dephasing (figure 7.b). Since it appears between two successive collisions on impurities, and since the phase mismatch between trajectories has to be smaller than 2π , one sees that the volume of this object is of order $\lambda_F^{d-1}l_e$.

It is important to evaluate the probability of occurrence of such quantum crossings because it will be shown to be a measure of the importance of quantum effects. Since the volume of a quantum crossing is of order $\lambda_F^{d-1}l_e$, a Diffuson propagating during a time t can be seen as an effective object of length $\mathcal{L} = v_F t$ and of "cross section" λ_F^{d-1} . Thus, it has a finite volume $v_F \lambda_F^{d-1} t$. The probability $dp_\times(t)$ of crossing of two Diffusons after a time dt in a volume $\Omega = L^d$ is thus proportional to the ratio between the volume of a Diffuson and the volume of the system :

$$dp_\times(t) = \frac{\lambda_F^{d-1} v_F dt}{\Omega} \propto \frac{1}{g} \frac{dt}{\tau_D} \quad (29)$$

where we have used (24) to introduce the dimensionless conductance g . Consider now an open system coupled to reservoirs. The time needed to travel throughout the sample is the Thouless time $\tau_D = L^2/D$. The probability of quantum crossing during this time is given by

$$p_\times(\tau_D) = \int_0^{\tau_D} dp_\times(t) = \frac{\lambda_F^{d-1} v_F \tau_D}{\Omega} \simeq \frac{1}{g} . \quad (30)$$

This is exactly the inverse conductance ! I believe that this is the most important message to understand phase coherence effects in disordered systems. All these effects can be simply understood in terms of quantum crossings and the probability of such crossings which measures the importance of quantum mechanical effects is simply given by the inverse of the dimensionless conductance g .

In a good metal, the conductance g is large, the volume of the tube is small, electrons do not spend much time in the system and quantum effects are very small. In the opposite limit, when g becomes of order 1, the volume of this tube is of the order of the volume of the system. It is so big that the probability of quantum crossing is of order 1. This corresponds to the

Anderson regime where electronic waves are localized by strong disorder. Here we shall not consider this regime but only the small disorder regime where quantum effects remain small. The approach to Anderson localization can be viewed as the proliferation of quantum crossings.

As a first qualitative but important conclusion of our discussion, we see that classical transport is described by a conductance $G = sge^2/h$. Quantum corrections are smaller than classical terms by a ratio $1/g$. This tells us immediately that the quantum corrections are of order G/g , that is e^2/h !

6 Weak localization

6.1 Weak localization and quantum crossings

We have seen that the classical probability and the conductance can be expressed as a sum of contributions of pairs of complex conjugated trajectories. Since trajectories can have quantum crossings, they can form closed loops (figure 8). It turns out that in such a loop (whose contribution is not included in eq. 13), the trajectories are *time-reversed*. One trajectory j and its time reversed j^T go in opposite directions. If there is time-reversal symmetry, they have the same action and thus they have exactly the same phase. This phase can be quite complicated because it depends on the disorder configuration but it is the same for both trajectories. So the contribution of these loops does not cancel on average. If the end points are far away like in fig. 8, the contribution of these new trajectories is small, of order $1/g$, but it leads to an experimentally observable effect : the weak localization correction to the conductance. This is a phase coherent effect because only trajectories of size smaller than the phase coherence length L_ϕ contribute to this additional contribution.

At this point, I want to stress that many presentations of weak localization correction emphasize the existence of a loop of opposite trajectories, but do not insist on the structure of the quantum crossing. This is rather quite important, because this is what explains the amplitude $1/g$ of the correction. This is where phase coherence is lost.

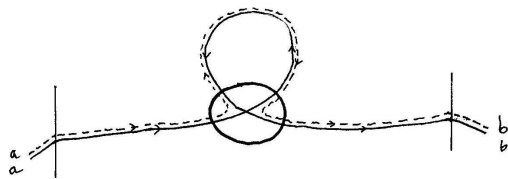


Fig. 8. Trajectory with a quantum crossing and a loop. In the loop, the two propagations are time reversed.

Using the same type of argument as in the previous section, let us evaluate the probability to have a loop for a trajectory which travels through the sample. Since there is a quantum crossing, the probability is small, of order $1/g$. Moreover, it depends on the distribution of loops in the disordered system. Let us call it $P_{int}(t)$. For the probability of traversing the sample with a loop, we have :

$$p_o(\tau_D) = \int_0^{\tau_D} P_{int}(t) dp_{\times}(t) = \frac{1}{g} \int_0^{\tau_D} P_{int}(t) \frac{dt}{\tau_D} . \quad (31)$$

We have also to remember that, because of decoherence in the loops, only those with time t smaller than τ_ϕ contribute. The resulting probability to have trajectories with loops of time smaller than τ_ϕ is :

$$p_o(\tau_\phi) = \int_0^{\min(\tau_D, \tau_\phi)} P_{int}(t) dp_{\times}(t) = \frac{1}{g} \int_0^{\min(\tau_D, \tau_\phi)} P_{int}(t) \frac{dt}{\tau_D} \quad (32)$$

where $P_{int}(t)$ is the probability to have loops of time t . This leads to a relative correction to the conductivity (or to the conductance) given by

$$\frac{\Delta\sigma}{\sigma_0} = \frac{\Delta G}{G_0} = -p_o(\tau_\phi) \quad (33)$$

The sign of the correction is negative because the trajectories j and j^T have opposite momenta. This quantum correction to the classical Drude conductivity is called the *weak localization* correction [10, 11, ?]. The phase coherence is broken by the coupling of the electrons to other degrees of freedom or due to electron-electron interactions. Such coherence breakdown is temperature dependent and can be phenomenologically described by a temperature dependent phase coherence length $L_\phi(T) = \sqrt{D\tau_\phi(T)}$: trajectories larger than L_ϕ do not contribute to the weak localization correction.

As we have seen above, the amplitude of the correction is proportional to $P_{int}(t)$, the distribution of loops. This quantity is nothing but the *return probability* after a time t . It is not exactly the classical return probability, which is the product of an amplitude with its complex conjugate (Diffuson). Here it corresponds to the product of an amplitude with the complex conjugate *time-reversed* amplitude. This object is named a "*Cooperon*". For closed trajectories, and with time-reversal symmetry, the two contributions, Cooperon and Diffuson, are equal. The return probability is thus doubled due to quantum coherence.

Eqs. (32, 33) have a meaning only in the diffusive regime for which $t > \tau_e$ (otherwise a loop cannot be formed). The contribution of the return probability has thus to be integrated between τ_e , the smallest time for diffusion, and the phase coherence time τ_ϕ . Replacing the bounds by exponential cutoffs, the weak localization correction can be cast in the form [13] :

$$\Delta G = -2s \frac{e^2}{h} \int_0^\infty P_{int}(t) (e^{-t/\tau_\phi} - e^{-t/\tau_e}) \frac{dt}{\tau_D} . \quad (34)$$

In order to evaluate ΔG in various situations, we now study the diffusion equation and its solutions.

6.2 How to solve the diffusion equation

In order to perform practical calculations, we have to calculate this distribution of closed trajectories, that is, the return probability. We have to solve a diffusion equation for this interference part. It looks very much like a classical diffusion equation (15), but there is an important difference. To account for magnetic field effects, it should be replaced by (in time representation) :

$$\left[\frac{\partial}{\partial t} - D(\nabla + \frac{2ie\mathbf{A}}{\hbar c})^2 \right] P(\mathbf{r}, \mathbf{r}', t) = \delta(\mathbf{r} - \mathbf{r}')\delta(t) . \quad (35)$$

The effect of the field is described by a covariant derivative (like in a Schrödinger equation) with an effective charge $2e$ to account for the pairing of trajectories.

To solve this equation (35), let us notice that it is a Green's equation, whose solutions are

$$P(\mathbf{r}, \mathbf{r}', t) = \theta(t) \sum_n \psi_n^*(\mathbf{r}) \psi_n(\mathbf{r}') e^{-E_n t} , \quad (36)$$

where $\theta(t)$ is the step function and $\{E_n, \psi_n\}$ are the eigenvalues and eigenfunctions of the eigenvalue equation associated to (35) :

$$-D(\nabla_{\mathbf{r}} + \frac{2ie\mathbf{A}}{\hbar c})^2 \psi_n(\mathbf{r}) = E_n \psi_n(\mathbf{r}) . \quad (37)$$

From eq. (36), we find that the integrated probability $P(t)$ has the simple form :

$$P(t) = \theta(t) \sum_n e^{-E_n t} . \quad (38)$$

This important result tells us that in order to evaluate the weak localization correction in any geometry, we simply need the eigenvalues of the diffusion equation in the corresponding geometry. We consider now a few examples.

6.3 Dimension dependence of the weak localization correction

Consider an infinite system, or with size $L \gg L_\phi$. For free diffusion in infinite space, the eigenvalues E_n of the diffusion equation are Dq^2 and the return probability $P(t)$ is given by (20). Since $P(t)$ is dimension dependent, we see that this weak localization correction depends dramatically on the space

dimensionality. Inserting (20) in (34), and writing $\tau_\phi = L_\phi^2/D$, we obtain the well-known results :

$$\Delta g = \begin{cases} -\frac{L_\phi}{L} & \text{quasi-1d} \\ -\frac{1}{\pi} \ln \frac{L_\phi}{l_e} & d = 2 \\ -\frac{1}{2\pi} \left(\frac{L}{l_e} - \frac{L}{L_\phi} \right) & d = 3 \end{cases} \quad (39)$$

Since $L_\phi(T)$ varies as a power-law with temperature, we obtain in particular the famous $\ln T$ dependence of the weak localization correction in $2d$.

It should be noticed that these results are meaningful if the correction stays smaller than the classical conductance (which can be written in the general form $g = A_d(k_F L/2\pi)^{d-1} l_e/L$). This defines a characteristic length ξ given by $\Delta g(\xi) \simeq g$, for which the weak-disorder perturbative regime breaks down. In $1d$ and $2d$, it is given by

$$\xi_{1d} = 2l_e \quad , \quad \xi_{2d} = l_e e^{\pi k_F l_e/2} \quad , \quad (40)$$

and for a quasi- $1d$ system $\xi_{q1d} \simeq M l_e$ where M is the number of channels. ξ is the localization length. For a review on the strong localization regime, see for example [14].

6.4 Finite systems, boundary conditions

In a mesoscopic system, the cutoff time in (32) is provided by τ_D . In other words, the cutoff length in eqs. (39) is now the size L of the system instead of L_ϕ . Therefore, from (39), we see that in quasi- $1d$, the weak localization correction is universal in the sense that it is a number, independent of disorder strength (l_e). In $2d$ and $3d$, the integral (34) diverges at small time and is cut off by τ_e , so that the correction cannot be universal [1].

In order to calculate quantitatively the weak localization correction and the return probability in a finite system, one must be careful to account properly for correct boundary conditions.

If the system is closed, electrons stay inside the system, so that $P(t) \xrightarrow[t \rightarrow \infty]{} 1$. The correct boundary condition is that the probability current vanishes at the boundary (Neumann condition). Therefore $q = n\pi/L$ with $n = 0, 1, 2, 3, \dots$. For $t \rightarrow \infty$, the contribution of the zero mode in eq. (38) gives correctly $P(t) \rightarrow 1$.

If the sample is perfectly connected to leads, electrons can leave the sample and $P(t) \xrightarrow[t \rightarrow \infty]{} 0$. The probability at the boundary has to vanish (Dirichlet boundary condition), because if it goes in the leads it never comes back in the same state. The zero mode is now excluded, $q = n\pi/L$ with $n = 1, 2, 3, \dots$.

$P(t) \rightarrow 0$ when $t \rightarrow \infty$ since the particle leaves the box at large time. Inserting the expression (38) of the return probability with $E_n = n^2 E_c$ into (34) gives immediately, in the limit $L_\phi \rightarrow \infty$:

$$\Delta g = -2 \sum_{n \neq 0} \frac{1}{\pi^2 n^2} = -\frac{1}{3} . \quad (41)$$

This result is proper to the perfectly connected wire.

6.5 Magnetic field effects

Ring or cylinder geometry : Sharvin-Sharvin oscillations

Consider first the geometry of a ring pierced by a Aharonov-Bohm flux. In the presence of the flux, each closed trajectory accumulates an Aharonov-Bohm phase $2\pi\phi/\phi_0$, where ϕ is the flux through the ring. The time-reversed trajectory accumulates an opposite phase $-2\pi\phi/\phi_0$, so that the relative phase shift between the two trajectories is $4\pi\phi/\phi_0$. The fact that this relative phase between the two time-reversed trajectories is twice the phase enclosed by one trajectory is the reason why average quantities oscillate with period $\phi_0/2 = h/2e$.

We need to calculate the return probability in this geometry. This can be done directly by solving (35). Here let us proceed by simple arguments. Remember that in a $1d$ infinite space, the probability to go from \mathbf{r} to \mathbf{r}' is given by (19) with $d = 1$. The return probability is obtained by writing $\mathbf{r} = \mathbf{r}'$. On a ring, this would be the return probability without making a loop, $1/\sqrt{4\pi Dt}$. The return probability after one loop of perimeter L necessarily contains a term $e^{-L^2/4Dt}$. The accumulated phase is $4\pi\phi/\phi_0$ so that the flux dependent contribution of trajectories making one loop is modulated by $\cos 4\pi\phi/\phi_0$. Adding together the contributions of m loops, we get simply the Fourier expansion of the flux dependent return probability :

$$P_{int}(t, \phi) = \frac{L}{\sqrt{4\pi Dt}} \sum_{m=-\infty}^{+\infty} e^{-m^2 L^2/4Dt} \cos 4\pi m \phi / \phi_0 . \quad (42)$$

Each harmonics of this expansion represents the return probability after m loops around the ring. Inserting this expression into (34) and after time integration, we obtain easily

$$\Delta G(\phi) = -s \frac{e^2}{h} \frac{L_\phi}{L} \left(1 + 2 \sum_{m=1}^{+\infty} e^{-mL/L_\phi} \cos 4\pi m \phi / \phi_0 \right) \quad (43)$$

which can be resumed to obtain

$$\Delta G(\phi) = -s \frac{e^2}{h} \frac{L_\phi}{L} \frac{\sinh L/L_\phi}{\cosh L/L_\phi - \cos 4\pi\phi/\phi_0} . \quad (44)$$

The harmonics decay exponentially with their order, since they correspond to longer and longer diffusive trajectories.

For a cylinder, there is a possibility for the electrons to diffuse along the z axis of the cylinder, so that (42) is simply multiplied by $L_z/\sqrt{4\pi Dt}$. Inserting this new probability into eq. (34), we obtain

$$\Delta G(\phi) = -s \frac{e^2}{\pi h} \frac{L}{L_z} \left[\ln \frac{L_\phi}{l_e} + 2 \sum_{m=1}^{+\infty} K_0(mL/L_\phi) \cos 4\pi m\phi/\phi_0 \right] \quad (45)$$

where K_0 is a modified Bessel function [15]. The $m = 0$ term is the usual 2d result (39). The contributions of the harmonics decay as e^{-L/L_ϕ} . These oscillations, predicted by Altshuler, Aronov and Spivak, were observed by Sharvin and Sharvin in 1981 [4, 16].

2d gas in a magnetic field

In the ring geometry, all pairs of diffusive trajectories would pick the same phase $4\pi\phi/\phi_0$. In a uniform magnetic field, small and large trajectories accumulate different fluxes $\phi(\mathcal{A}) = B\mathcal{A}$ depending on their area \mathcal{A} . So the return probability is balanced by the average $\langle \cos 4\pi\phi(\mathcal{A})/\phi_0 \rangle_{\mathcal{A}}$ on the distribution of areas \mathcal{A} formed by the time-reversed diffusive trajectories.

Let us start with a qualitative evaluation. Short trajectories accumulate a flux smaller than the flux quantum and their contribution survives. Large trajectories accumulate flux larger than ϕ_0 and their contribution vanishes. When the magnetic field increases, the contribution of smaller and smaller trajectories is progressively suppressed. Trajectories smaller than some field dependent characteristic length L_B corresponding to $BL_B^2 \simeq \phi_0$ will not contribute. To this length L_B corresponds a characteristic time $\tau_B = L_B^2/D \simeq \phi_0/B D$, so that we can expect

$$\langle \cos 4\pi\phi/\phi_0 \rangle_{\mathcal{A}} \simeq e^{-t/\tau_B} \quad . \quad (46)$$

Trajectories which enclose more than one flux quantum do not contribute to the return probability. Because of this new cutoff time, we can expect a field dependence of the weak localization of the form

$$\Delta g = -\frac{1}{\pi} \ln \frac{\min(L_\phi, L_B)}{l_e} \quad (47)$$

instead of (39).

The exact calculation is straightforward starting from eq. (38). The eigenvalues E_n are solutions of an effective Schrödinger for a free particle of mass $m = \hbar/2D$ and charge $-2e$ in a uniform field B . They are precisely the Landau levels, namely

$$E_n = (n + \frac{1}{2}) \frac{4eDB}{\hbar} , \quad (48)$$

where n is an integer. The degeneracy of these levels for an area S is $g_n = \frac{2eB}{\hbar} S$. The integrated return probability $P_{int}(t)$ is just given by the sum $\sum_n g_n e^{-E_n t}$, that is :

$$P_{int}(t, B) = \frac{BS/\phi_0}{\sinh(4\pi BDt/\phi_0)} \quad (49)$$

where $\phi_0 = h/e$ is the flux quantum. This expression is nothing but the partition function of the harmonic oscillator. In the limit $B \rightarrow 0$, one recovers the result for free diffusion : $S/(4\pi Dt)$. For large times, $P_{int}(t, B)$ decreases exponentially with the characteristic time $\tau_B = \phi_0/4\pi BD$ introduced qualitatively in (46). It describes the dephasing of time reversed trajectories. Inserting eq. (49) in (34), and performing the integral, we get :

$$\Delta g(B) = -\frac{1}{2\pi} \left[\Psi \left(\frac{1}{2} + \frac{\hbar}{4eDB\tau_e} \right) - \Psi \left(\frac{1}{2} + \frac{\hbar}{4eDB\tau_\phi} \right) \right] \quad (50)$$

where Ψ is the digamma function. This expression corresponds to the approximation (47). The weak localization correction is negative and cancelled by the magnetic field. As a result, a negative magnetoresistance is a well-known signature of weak localization (figure 9). A magnetoresistance measurement is a very interesting and useful tool to estimate τ_ϕ . The correction cancels when $\tau_B \simeq \tau_\phi$, that is for a field B_ϕ corresponding to a flux quantum through an area $B_\phi L_\phi^2$. Doing the same measurement at different temperatures is the usual method to extract $\tau_\phi(T)$.

7 Conductance fluctuations

7.1 Universality as a signature of quantum coherence

At a scale $L < L_\phi$, a conductor is a quantum object. Its conductance depends on the interference pattern between all diffusive trajectories. This interference pattern can be modulated by external parameters, like a magnetic field or a gate voltage. For example, figure 10.a represents the variation of the conductance with the magnetic field, performed for 46 different samples (actually the same sample which has been annealed several times, so that the impurity configuration has changed and the interference pattern is different). It exhibits "fluctuations" which are reproducible for a given configuration of disorder. They are a "*fingerprint*" of this configuration. Figure 10.b displays the average conductance, obtained by an average over the 46 samples. One clearly sees the weak localization correction, which is destroyed around some characteristic field B_ϕ . Interestingly, for the same characteristic field, the variance of the fluctuations, displayed on figure 10.c is reduced by a factor 2.

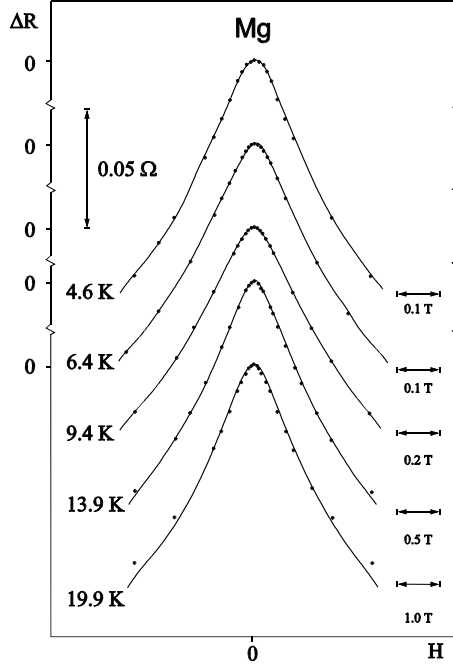


Fig. 9. Magnetic field dependence of the magnetoresistance of a Mg film, for different temperatures. The points are experimental results and the solid curves correspond to (50). The time $\tau_\phi(T)$ is a fitting parameter [11].

Universality of the conductance fluctuations is a signature of quantum transport. Classically, one would expect the system to be considered as an addition of large number of incoherent elements. This number is of order $\mathcal{N} = (L/L_c)^d$, where L_c would be a correlation length, of the order of the mean free path. Then one would expect relative fluctuations of the conductance G of order of

$$\frac{\delta G}{G} \simeq \frac{1}{\sqrt{\mathcal{N}}} = \left(\frac{L_c}{L} \right)^{d/2} \quad (51)$$

so that, since G varies as L^{d-2} :

$$\delta G \propto L^{\frac{d-4}{2}} \quad , \quad (52)$$

and vanishes for large L . The system is said to be self-averaging. But the fact that the fluctuation δG stays actually finite means that there are strong correlations due to quantum coherence. Moreover, if one considers fully coherent ($L < L_\phi$) systems with quite different conductances, a good metal, a bad metal, or a semiconductor, one finds that the amplitude of the "oscillations" is always the same : it does not depend on the disorder. It is universal,

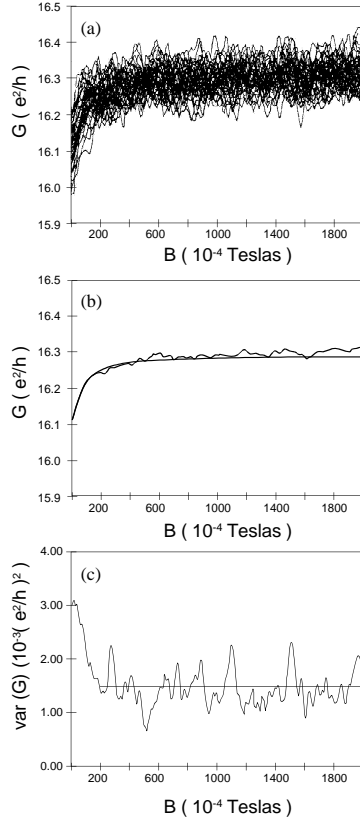


Fig. 10. Reproducible fluctuations of the magnetoconductance in units of e^2/h , at $T = 45\text{mK}$ for Si doped GaAs. Fig. a shows 46 plots as function of the magnetic field, for the same sample after successive annealing. Each plot corresponds to a disorder configuration and is called a magnetofingerprint. The amplitude of the fluctuations is smaller than e^2/h because $L > L_\phi$ (see eq. 66). Fig. (b) presents the average conductance versus field. The weak localization correction disappears beyond a characteristic field. Above the same field, the variance of the fluctuations displayed on fig. (c) is divided by a factor 2, corresponding to the destruction of the Cooperon [17].

of order e^2/h . *A priori*, we are not so surprised that these fluctuations are universal since our simple argument of section 5 showed that all quantum effects have to be of order e^2/h .

7.2 conductance fluctuations and speckle correlations in optics

Here it is quite useful to compare the physics of electronic transport with similar physics in optics where one measures the fluctuations of a transmission coefficient. In optics the light scattered by a diffusing medium forms a *speckle* pattern on a screen, and we want to describe the speckle fluctuations (This is exactly a generalization of Young experiments. Two slits produce well defined fringes. Here the diffusing medium, like *e.g.* colloidal suspension, produces a complicated pattern called a speckle). A laser beam is sent on a diffusing medium along an incident direction a and the diffused intensity is measured along a direction b . The speckle pattern displayed on figure 11 represents the intensity measured along a direction b for a fixed incident direction a . So a given intensity on the screen represents the transmission coefficient T_{ab} from

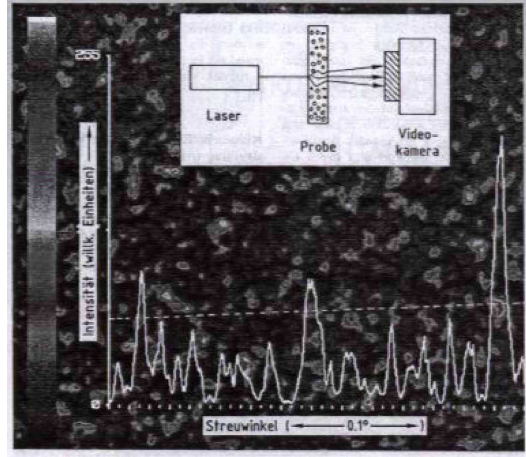


Fig. 11. A typical speckle pattern. The white and noisy curve represents the angular dependence of the light intensity along the cut represented by the dashed line. The relative fluctuations are of order unity (courtesy of G. Maret).

an incident direction a to an emergent direction b . We notice immediately that there are black spots, meaning that the relative fluctuations of this coefficient are of order 1. This is the Rayleigh law :

$$\overline{\delta T_{ab}^2} = \overline{T_{ab}}^2 \quad . \quad (53)$$

In electronics however, relative fluctuations of the conductance are very small, of order $1/g^2$. Here we want to understand why fluctuations are large in optics and small in electronics, namely to compare conductance fluctuations and fluctuations of transmission coefficient.

A very convenient way to link the two fields of optics and electronics is to use the Landauer formalism, which explicitly expresses the conductance as a transmission coefficient. This is formalized by Landauer formula $g = \sum_{ab} T_{ab}$ (25). The main difference between optics and electronics is that in optics, it is possible to measure each transmission coefficient T_{ab} while in electronics the conductance is related by eq. (25) to a *sum* over all incoming and outgoing channels.

The average transmission is a probability. It is a sum of contributions of paired trajectories. Assuming that the angular (or channel) dependence of $\overline{T_{ab}}$ is negligible, all the channels contribute equally to the conductance, so that, from (25) :

$$\overline{G} = \frac{e^2}{h} M^2 \overline{T_{ab}} \quad (54)$$

and the average transmission coefficient is thus equal to

$$\overline{T}_{ab} = \frac{g}{M^2} . \quad (55)$$

Now we want to calculate the correlation between two transmission coefficients, that is the function $\overline{T_{ab}T_{a'b'}}$. This quantity is the product of two average transmission coefficients plus a correlation term :

$$\overline{T_{ab}T_{a'b'}} = \overline{T}_{ab} \overline{T}_{a'b'} + \overline{\delta T_{ab} \delta T_{a'b'}} . \quad (56)$$

The correlation term is constructed by pairing of trajectories corresponding to different transmission coefficients (Figure 12).

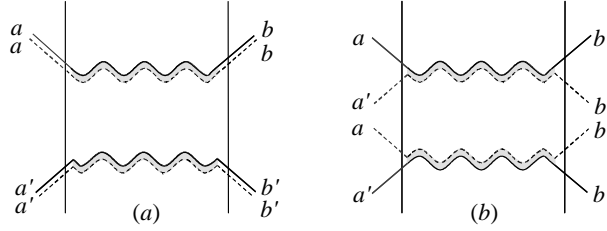


Fig. 12. Schematic representation of the two contributions to the product $\overline{T_{ab}T_{a'b'}}$. The first (a) corresponds to the product $\overline{T}_{ab} \overline{T}_{a'b'}$. The second (b) gives a contribution to the correlation function that we shall denote by $\overline{\delta T_{ab} \delta T_{a'b'}}^{C_1}$.

One sees that there is no dephasing between the diffusive paired trajectories, except outside the sample since a and a' (b and b') may correspond to different incoming (outgoing) channels. The second term is therefore of the form $\overline{T}_{ab}^2 f(a, a', b, b')$ where f is a short range function which vanishes rapidly as soon as $a \neq a'$ or $b \neq b'$. If a and b are angular directions, f is a rapidly decreasing function of the angles. In a wave guide geometry, where the modes are quantized, $f = \delta_{aa'}\delta_{bb'}$. This contribution to the correlation function $\overline{\delta T_{ab} \delta T_{a'b'}}$ is called C_1 :

$$\overline{\delta T_{ab} \delta T_{a'b'}}^{C_1} = \overline{T}_{ab}^2 \delta_{aa'}\delta_{bb'} . \quad (57)$$

For $a = a'$ and $b = b'$, we obtain the Rayleigh law (53), that is

$$\overline{T}_{ab}^2 = 2 \overline{T}_{ab}^2 . \quad (58)$$

The amplitude of the fluctuations is of the order of the average. This explains why there are black spots on figure 11.

In order to calculate the conductance fluctuations, we have to sum over all incoming and outgoing channels :

$$\overline{\delta g^2} = \sum_{aa'bb'} \overline{\delta T_{ab} \delta T_{a'b'}} = \sum_{aa'bb'} \overline{T}_{ab}^2 \delta_{aa'}\delta_{bb'} = M^2 \overline{T}_{ab}^2 = \frac{g^2}{M^2} \ll 1 . \quad (59)$$

The sum is small since most of the terms are negligible. So our picture explains the important fluctuations of T_{ab} but *not* the amplitude of the conductance fluctuations. This means that additional contributions to the correlation function $\overline{\delta T_{ab} \delta T_{a'b'}}$ may have been forgotten.

A next contribution is obtained by pairing trajectories in a different way. One possibility is to exchange the quantum amplitudes, and to have one crossing as shown in figure 13.a. This contribution is smaller by a factor $1/g$, so that its contribution to the correlation function $\overline{\delta T_{ab} \delta T_{a'b'}}$ is small. But it has an angular dependence different from the previous contribution. Figure 13.a shows that there is a phase factor either for the incoming or the outgoing beam, so that instead of (57), we have for this second contribution, usually called C_2 ,

$$\overline{\delta T_{ab} \delta T_{a'b'}}^{C_2} = \frac{2}{3g} \overline{T_{ab}}^2 (\delta_{aa'} + \delta_{bb'}) . \quad (60)$$

The factor $2/3$ results from an integration over the position of the quantum crossing [1]. A sum over all incoming and outgoing channels gives for this contribution :

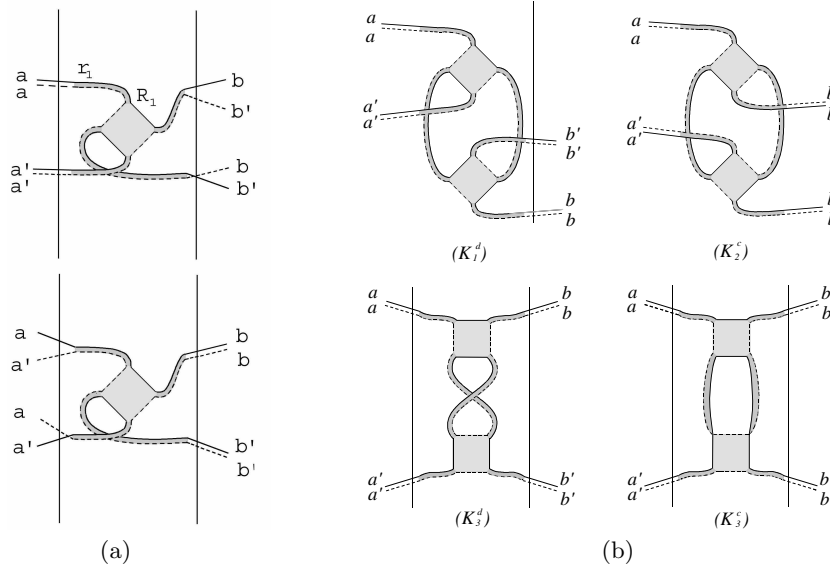


Fig. 13. Schematic representation of the C_2 and C_3 contributions to the correlation function $\overline{\delta T_{ab} \delta T_{a'b'}}$. κ_1^d and κ_3^d are Diffuson contributions. κ_2^c and κ_3^c are Cooperon contributions.

$$\overline{\delta g^2} = \sum_{aa'bb'} \frac{2}{3g} \overline{T_{ab}}^2 (\delta_{aa'} + \delta_{bb'}) = \frac{4g}{3M} \ll 1 . \quad (61)$$

Still this contribution cannot explain the amplitude of the observed conductance fluctuations, since it vanishes in the large M limit. Let us consider the next contribution shown on figure 13.b, with two quantum crossings. We see that this so-called C_3 contribution is smaller than the first one by a factor $1/g^2$:

$$\overline{\delta T_{ab} \delta T_{a'b'}}^{C_3} = \frac{2}{15g^2} \overline{T_{ab}}^2 \quad (62)$$

The $2/15$ factor comes from the integration over the positions of the two quantum crossings [1] (see next section). This term has *no angular dependence*, so that the summation over channels contains now M^4 terms and the contribution to the conductance fluctuations is

$$\overline{\delta g^2} = \frac{2}{15} . \quad (63)$$

The fluctuations are universal, independent on the strength of disorder. In summary, they are universal since the corresponding correlation function is constructed with two conductances and two quantum crossings, giving $g^2 \times 1/g^2 \simeq 1$. The contributions with no crossing or one crossing cancel because of angular dependences. The next terms with n crossings are negligible, of order $1/g^{n-2}$. In optics when one considers a speckle pattern, the first contribution is the most important, and the one crossing and two crossings contributions are very difficult to observe (They can be observed since, although very small, they are a different angular dependence, and also different temporal or frequency dependences [1]). In electronics, only the third contribution with two crossings is important after summation over incoming and outgoing channels. In summary, $\sum_{ab} T_{ab}$ has much smaller fluctuations than T_{ab} . We have now a simple recipe to evaluate average quantities or correlations functions : each quantum crossing gives a factor $1/g$.

7.3 Amplitude of the conductance fluctuations

In order to calculate quantitatively the conductance fluctuations, their dependence on geometry or external parameters, we must analyze more precisely the structure of the paired trajectories in figure 13.b. In addition to the crossings, there is a loop. And we have to integrate on the distribution of loops, like for the weak localisation correction. In contrast with the weak localization correction, this loop is formed by two crossings instead of one. So here for a given position of one crossing, we have to integrate on the position of the second crossing. Since this second crossing is necessarily along the loop formed by the two crossings, the integration over the position of the second crossing gives a volume element proportional to the length of the loop. For a trajectory of length $v_F t$, the volume is $v_F t \lambda_F^{d-1}$. Moreover, $P(t)$ contains two-Diffusons and two-Cooperons contributions. A careful examination of the possible crossing and trajectories shows the only possible diagrams shown on

figure 13. Taking into account their degeneracy [1], we obtain an expression which is as simple as the weak localization correction :

$$\overline{\delta G^2} = 6s^2 \left(\frac{e^2}{h} \right)^2 \int_0^\infty t [P_{cl}(t) + P_{int}(t)] e^{-t/\tau_\phi} \frac{dt}{\tau_D^2} \quad (64)$$

There is an equal contribution of loops with Diffusons or Cooperons. In a magnetic field, the Cooperon contribution is suppressed so that the variance is reduced by a factor 2, as seen on fig. 10.c. This happens for the same magnetic field B_ϕ as the destruction of the weak localisation correction, that is for a flux quantum through the system or through L_ϕ^2 .

Incoherent processes not only destroy the Cooperon contribution but also the Diffuson contribution. This could appear surprising since we have seen that this Diffuson contribution corresponds to classical diffusion, and therefore should not be phase sensitive. However, this Diffuson contribution is not really the classical contribution, since it is constructed by pairing trajectories corresponding to *different* realizations of the system. If there is a phase breaking event, it affects equally one amplitude and its complex conjugate. But here, the phase breaking event may affect differently the two amplitudes since they correspond to different systems.

Like we have done above for the weak localization correction (34), we can now evaluate quite easily $\overline{\delta G^2}$ given by (64) for different geometries from the corresponding expression of the return probability $P(t)$. Let us do it for a quasi-1d mesoscopic wire. $P(t)$ is given by

$$P(\mathbf{q}, t) = \sum e^{-Dq^2 t}$$

where $q = n\pi/L$ is quantized by the Dirichlet boundary conditions corresponding to a perfectly connected wire (no zero mode, see section 6.4). Inserting $P(t)$ in (64), we get

$$\overline{\delta g^2} = 6 \sum_{n>0} \frac{1}{\pi^4 n^4} = \frac{2}{15} . \quad (65)$$

Let us remark that the choice of the boundary conditions is very important. If the system were closed or poorly connected, corresponding to Neumann boundary conditions, then the contribution of the zero mode would lead to a divergence, or at least non universality of the fluctuations.

In the macroscopic limit, when $L_\phi \gg L$, we can treat the system as infinite, replace $P(t)$ by its dependence (20) for an infinite system $(\tau_D/4\pi t)^{d/2}$ and multiply by the exponential decay e^{-t/τ_ϕ} in the integral. Then we obtain :

$$\overline{\delta g^2} \propto \left(\frac{L_\phi}{L} \right)^{\frac{4-d}{2}} . \quad (66)$$

We are not surprised by this result. It is exactly the one anticipated from our simple argument (51) treating the fluctuations as due to incoherent contributions of correlated regions of size L_ϕ .

Finally, it is quite easy to notice, that correlations functions of n conductances imply $2n - 2$ crossings. Therefore the n^{th} cumulant of the conductance distribution is of order $g^n/g^{2n-2} \propto 1/g^{n-2}$. It vanishes for $n > 2$ in the metallic limit $g \rightarrow \infty$, so that the conductance distribution is indeed gaussian.

8 Diffusion on graphs and spectral determinant

The calculation of the weak localization correction or of the conductance fluctuations, as well as of other thermodynamical quantities like orbital magnetic susceptibility [1, 18] can be extended to the case of any structure – called a network – made of quasi- one-dimensional diffusive wires. First, we note that the quantities of interest, like the weak localization correction (34) or the conductance fluctuations (64), have the same structure :

$$\int t^\alpha P(t) e^{-\gamma t} dt , \quad (67)$$

where $\gamma = 1/\tau_\phi$. From (38), the time integral of $P(t)$ can be straightforwardly written in terms of a quantity called the *spectral determinant* $S(\gamma)$:

$$\int_0^\infty dt P(t) e^{-\gamma t} = \sum_n \frac{1}{E_n + \gamma} = \frac{\partial}{\partial \gamma} \ln S(\gamma) , \quad (68)$$

where $S(\gamma)$ is, within a multiplicative constant independent of γ :

$$S(\gamma) = \prod_n (\gamma + E_n) , \quad (69)$$

E_n being the eigenvalues of the diffusion equation (37). Using standard properties of Laplace transforms, the above time integrals can be rewritten in terms of the spectral determinant, so that the weak localization to the conductivity and the conductivity fluctuations respectively read :

$$\Delta\sigma = - 2s \frac{e^2}{h} \frac{D}{\Omega} \frac{\partial}{\partial \gamma} \ln S_{int}(\gamma) \quad (70)$$

$$\langle \delta\sigma^2 \rangle = - 6s^2 \frac{e^4}{h^2} \frac{D^2}{\Omega^2} \frac{\partial^2}{\partial \gamma^2} [\ln S_{cl}(\gamma) + \ln S_{int}(\gamma)] , \quad (71)$$

where S_{cl} and S_{int} are the spectral determinants associated respectively to the diffusion equation for the Diffuson and the Cooperon. Ω is the total volume of the system. These expressions are quite general, strictly equivalent to expressions (34,64). Their interest is that, on a network, the spectral determinant takes a very simple form. By solving the diffusion equation (35) on each

link, and then imposing Kirchoff type conditions on the nodes of the graph with N nodes, the problem can be reduced to the solution of a system of N linear equations relating the eigenvalues at the N nodes. Let us introduce the $N \times N$ matrix M :

$$M_{\alpha\alpha} = \sum_{\beta} \coth(\eta_{\alpha\beta}) \quad , \quad M_{\alpha\beta} = -\frac{e^{i\theta_{\alpha\beta}}}{\sinh \eta_{\alpha\beta}} \quad (72)$$

The sum \sum_{β} extends to all the nodes β connected to the node α ; $l_{\alpha\beta}$ is the length of the link between α and β . $\eta_{\alpha\beta} = l_{\alpha\beta}/L_{\phi}$. The off-diagonal coefficient $M_{\alpha\beta}$ is non zero only if there is a link connecting the nodes α and β . $\theta_{\alpha\beta} = (4\pi/\phi_0) \int_{\alpha}^{\beta} A \cdot dl$ is the circulation of the vector potential between α and β . N_B is the number of links in the graph. It can then be shown that the spectral determinant takes the very convenient form [1, 18] :

$$S = \left(\frac{L_{\phi}}{L_0} \right)^{N_B - N} \prod_{(\alpha\beta)} \sinh \eta_{\alpha\beta} \det M \quad (73)$$

L_0 is an arbitrary length independent of γ (or L_{ϕ}). We have thus transformed the spectral determinant which is an infinite product in a finite product related to $\det M$. Using (73), mesoscopic quantities (70,71) can be easily predicted for any geometry of diffusive networks [1, 18].

9 Interaction effects

Until now we have not considered the role of electron-electron interactions. They turn out to give small corrections to transport quantities like the average conductivity, but they play an important role to understand thermodynamic properties like persistent currents [1]. Moreover, until now we have introduced by hand a phase coherence time τ_{ϕ} (or length L_{ϕ}). We wish now to understand the microscopic origin for the loss of quantum coherence. This phase coherence is limited by the interactions with other degrees of freedom, in particular other electrons through their mutual interaction. We want to discuss now how e-e interactions break phase coherence.

On one hand, interaction effects can be considered as negligible. We know from Landau theory of Fermi liquids that in an interacting electron gas, free particles have simply to be replaced by *quasiparticles* which are dressed objects, screened by the cloud of other electrons. These quasiparticles have a long lifetime which diverges when approaching the Fermi level. From Landau, we know that

$$\frac{1}{\tau_{ee}(\epsilon)} \propto \epsilon^2 \quad (74)$$

where $\epsilon = E - E_F$ is the energy of the quasiparticle measured from the Fermi level. On the other hand, in a disordered metal, electrons move diffusively, that is very slowly. They spend long time close to each other. Qualitatively, we can expect that diffusion somehow enhance the effect of interactions. We may ask if and how expression (74) is changed because of the diffusion. Moreover, since the diffusive motion is dimensionality dependent, the modified lifetime should also depend on this dimensionality.

Interaction between electrons is expected to have two major effects :

- First, each electron is not only sensitive to the disordered potential but also to the fluctuations of the electronic density due to other electrons. This additional fluctuating potential modifies the position of the energy levels, especially near the Fermi level. So we may expect a modification in the repartition of the energy levels, that is a change of the density of states near the Fermi level. We shall show that the density of states exhibits a decrease, the so-called "Altshuler-Aronov" anomaly. This reduction of the density of states is accompanied by a reduction of the conductivity.
- Since the e-e interaction is a inelastic process, each quasiparticle has a finite lifetime which limits the phase coherent properties like weak localization, since the coherence between time-reversed trajectories is necessarily limited by this lifetime.

9.1 Screening

In order to describe interaction effects, let us start with a few reminders about the screening of interaction. The bare Coulomb interaction potential is $U_0(R) = e^2/R$, that is in $3d$, $U_0(q) = 4\pi e^2/q^2$. The screened Coulomb interaction $U(q)$ is given by

$$U(q) = \frac{U_0(q)}{1 + \chi_0(q)U_0(q)} \quad (75)$$

where, in the small q limit (we are interested in the diffusive regime where $ql_e \ll 1$), the susceptibility $\chi_0(q)$ is the Pauli susceptibility, that is the density of states $2\rho_0$. Therefore, the screened interaction is given by

$$U(q) = \frac{4\pi e^2}{q^2 + \kappa^2} \quad , \quad U(R) = \frac{e^2}{R} e^{-\kappa R} \quad , \quad (76)$$

where the Thomas-Fermi vector κ (inverse screening length) is

$$\kappa^2 = 8\pi e^2 \rho_0 \quad . \quad (77)$$

In the diffusive limit $ql_e \ll 1$, the screened interaction can be approximated by

$$U(q) = \frac{4\pi e^2}{\kappa^2} = \frac{1}{\chi_0} = \frac{1}{2\rho_0} \quad , \text{ that is } \quad U(R) = \frac{1}{2\rho_0} \delta(R) \quad . \quad (78)$$

On the scale of diffusion, the screened interaction can be considered as a local interaction.

However, it turns out that screening is not instantaneous since electrons have to diffuse to screen a local charge. Therefore the interaction is actually frequency dependent. This dynamical screening is described by the frequency dependent susceptibility $\chi_0(q, \omega)$ which accounts for the dynamical charge reorganization :

$$\chi_0(q, \omega) = 2\rho_0 \frac{Dq^2}{-i\omega + Dq^2} \quad . \quad (79)$$

Therefore eq. (78) becomes

$$U(q, \omega) = \frac{1}{\chi_0} = \frac{1}{2\rho_0} \frac{-i\omega + Dq^2}{Dq^2} \quad . \quad (80)$$

9.2 Density of states anomaly

As we have done above, we shall avoid technicalities of diagrammatic theory, and try to get the important results from qualitative arguments. Although we have not elaborated on the theory of Green's functions, let me remind you at least that the Green's function is related to the density of states by

$$\rho(\epsilon) = \frac{-1}{\pi\Omega} \int \text{Im} G_\epsilon(\mathbf{r}_0, \mathbf{r}_0) d\mathbf{r}_0 \quad . \quad (81)$$

As we have seen in section 3, $G(\mathbf{r}_0, \mathbf{r}_0)$ is the sum of contributions from all closed trajectories from \mathbf{r}_0 to \mathbf{r}_0 (figure 14.a). All these amplitudes have different different and random phases and their contribution cancels in average. What remains is the contribution of short trajectories, giving an average density of states $\bar{\rho}(\epsilon) = 2\rho_0$.

How can we effects of diffusion + interactions appear on the density of states? The non-interacting density of states (or Green's function) is a single electron property, and therefore involve single trajectories. In the presence of electron-electron interaction, each electron trajectory can be paired with the trajectory of a second electron, with which it interacts. Then by pairing these two trajectories, we can construct a Diffuson. More precisely, we pair an amplitude corresponding to one electron to the conjugate amplitude corresponding to another electron. Their interaction is represented by a wiggly line in Figure 14.b. There are actually two possible contributions, depending on the position of the interaction line. They are nothing but the Hartree and the exchange (Fock) contributions. We can conveniently separate the "diagrams" in three different parts :

- A short range part close to the point \mathbf{r}_0 . It resembles somehow to a quantum crossing, with a dephasing between the three trajectories,

- two long ranged Diffusons,
- an interaction region between \mathbf{r} and \mathbf{r}' .

So, we can easily construct the structure of this additional contribution to the density of states :

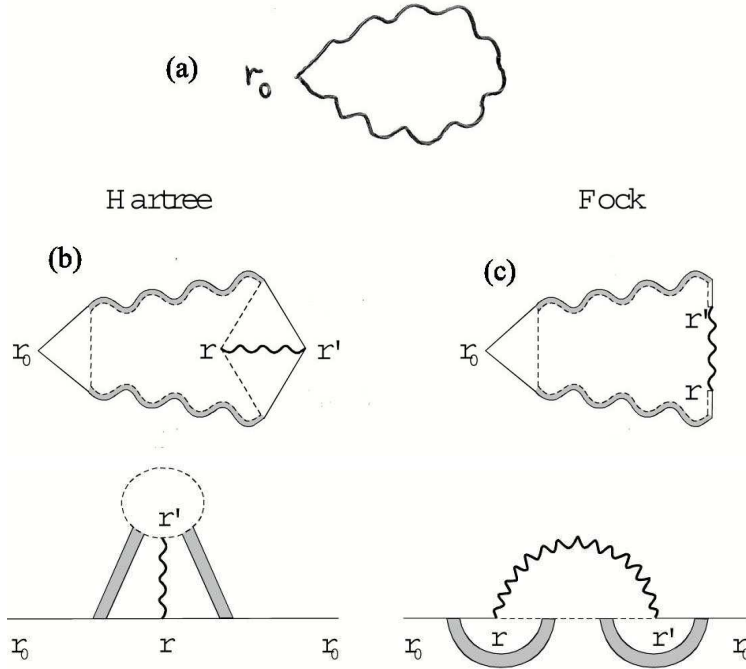


Fig. 14. a) Diagram for the non-interacting density of states. After disorder averaging, the contribution of diffusive trajectories vanishes because of their random phase. Hartree (b) and exchange (c Fock) diagrams for the density of states anomaly. These contributions survive disorder averaging since they contain paired trajectories. Upper : schematic representation exhibiting clearly the three regions, a short range region, diffusive trajectories and the interaction region. Bottom : usual diagrammatic representation. These two representations are equivalent.

$$\frac{\delta \rho}{\rho_0} \simeq -\frac{\lambda_\rho}{g} \int^{\hbar/\epsilon} P(t) \frac{dt}{\tau_D} . \quad (82)$$

λ_ρ is a dimensionless parameter which describes the strength of the interaction. It contains both the Hartree and exchange contributions. The $1/g$ reduction comes from the quantum crossing. And the distribution of loops formed by the paired trajectories is given by the return probability $P(t)$. For

an energy ϵ , the two amplitudes must be taken at different energies ω and $\omega - \epsilon$. Therefore the two trajectories can stay in phase only during a time $\Delta \simeq \hbar/\epsilon$, so that the upper cutoff in the integral is \hbar/ϵ .

Here our aim is simply to present the structure of the result without entering into details. Another qualitative derivation is given in [19], a detailed discussion is proposed in [1], and the original calculation is done in [20]. We see that the amplitude and the structure of this correction to the density of states looks very similar to that of the weak localization correction (34), except that the upper cutoff is not τ_ϕ but \hbar/ϵ . Unlike for the weak localization, this correction depends on the classical return probability, so that it is not suppressed by a magnetic field [21].

The form (82) is approximate. A more sophisticated calculation replaces the upper cutoff by a Fourier transform. Moreover since $g \propto \rho_0 \Omega / \tau_D$, eq. (82) becomes

$$\delta\rho(\epsilon) = -\frac{\lambda_\rho}{\pi\Omega} \int_0^\infty P(t) \cos \epsilon t dt \quad . \quad (83)$$

From the expression (20) of $P(t)$, we obtain the energy dependence of the density of states anomaly :

$$\delta\rho(\epsilon) \propto -\frac{\lambda_\rho}{D\Omega} \begin{cases} L_\epsilon & \text{quasi-1d} \\ \ln \frac{L_\epsilon}{l_e} & d=2 \\ \frac{1}{l_e} - \frac{1}{L_\epsilon} & d=3 \end{cases} \quad (84)$$

where the characteristic length L_ϵ is $L_\epsilon = \sqrt{\hbar D / \epsilon}$. The structure of this correction related to $P(t)$ is similar to the weak localization correction (39), except that the characteristic length L_ϕ has been replaced by L_ϵ . This reduction of the density of states can be observed experimentally by tunnel measurements, since it reflects as an anomaly in the voltage dependence of the tunnel conductance G_t . At zero temperature, the relative correction to the tunnel conductance is given by

$$\frac{\delta G_t}{G_t} = \frac{\delta\rho(\epsilon = eV)}{\rho_0} \quad . \quad (85)$$

At finite temperature, it is not difficult to generalize the expression (83) of the density of states anomaly as

$$\delta\rho(\epsilon, T) = - \int f'(\epsilon - \omega) \delta\rho(\omega) d\omega \quad , \quad (86)$$

where f' is the derivative of the Fermi function. After a Fourier transform, we find

$$\delta\rho(\epsilon, T) = -\frac{\lambda_\rho}{2\pi\Omega} \int_0^\infty R_T(t)P(t) \cos \epsilon t dt \quad , \quad (87)$$

where the thermal function $R_T(t)$ is given by $R_T(t) = \pi T t / \sinh \pi T t$. The temperature dependence of the tunnel conductance anomaly, also called zero-bias anomaly is

$$\frac{\delta G_t(V, T)}{G_t} = -\frac{1}{\rho_0} \int \delta\rho(\epsilon, T) f'(\epsilon - eV) d\epsilon \quad , \quad (88)$$

or, after a Fourier transform :

$$\frac{\delta G_t(V, T)}{G_t} = -\frac{\lambda_\rho}{2\pi\rho_0\Omega} \int_0^\infty R_T^2(t)P(t) \cos eVt dt \quad , \quad (89)$$

This correction has been measured for various systems with different dimensionalities and the $1/\sqrt{V}$, $\ln V$, and \sqrt{V} predicted by (84,89) respectively in 1, 2 and 3 dimensions have been observed [22].

9.3 Correction to the conductivity

Taking into account the interaction between electrons leads also to a reduction of the conductivity. Without going into the details of the calculations, we can argue that this reduction is a consequence of the correction to the density of states. Both effects result from the scattering of an electron by the charge fluctuations induced by disorder. The temperature dependence of the conductivity $\sigma(T)$ is related to its energy dependence at $T = 0K$ by $\sigma(T) = -\int f'(\epsilon)\sigma(\epsilon)d\epsilon$, where $f'(\epsilon)$ is the derivative of the Fermi function. Since the conductivity is proportional to the density of states (Einstein relation), we expect that the density of states anomaly leads to a correction of the conductivity given by

$$\frac{\delta\sigma(T)}{\sigma_0} = \int d\epsilon \left(-\frac{\partial f}{\partial \epsilon} \right) \frac{\delta\rho(\epsilon, T)}{\rho_0} \quad , \quad (90)$$

where σ_0 is the Drude conductivity (22). For a static interaction, the density of states correction $\delta\rho(\epsilon, T)$ is given by (87). Upon Fourier transforming, we have

$$\delta\sigma(T) = -\lambda_\sigma \left(\frac{e^2 D}{\pi\Omega} \right) \int_0^\infty R_T^2(t)P(t)dt \quad (91)$$

where λ_σ is a parameter which depends on the interaction [23]. Using the expression (20) of $P(t)$, we obtain the temperature dependence of the correction to the conductance

$$\delta g(T) \propto -\lambda_\sigma \begin{cases} \frac{L_T}{L} & \text{quasi-1d} \\ \ln \frac{L_T}{l_e} & d=2 \\ \frac{L}{l_e} - \frac{L}{L_T} & d=3 \end{cases} \quad (92)$$

where the thermal length L_T is defined by $L_T = \sqrt{\hbar D/T}$. In $2d$, the temperature dependence is logarithmic like the weak localization correction. Unlike the weak localization correction, this correction to the conductivity is not sensitive to a magnetic field. Therefore both corrections can be separated experimentally by the application of a magnetic field.

9.4 Lifetime of quasiparticle

Consider a Fermi sea and inject a quasiparticle in a state $|\alpha\rangle$ with energy ϵ_α above the Fermi sea. It interacts with another particle ($|\gamma\rangle, \epsilon_\gamma$) and the final state consists in two quasiparticles ($|\beta\rangle, \epsilon_\beta$) and ($|\delta\rangle, \epsilon_\delta$) above the Fermi sea (figure 15). Energy conservation implies $\epsilon_\alpha + \epsilon_\gamma = \epsilon_\beta + \epsilon_\delta$. The lifetime of the state $|\alpha\rangle$ is given by the Fermi golden rule and it is related to the matrix element of the interaction :

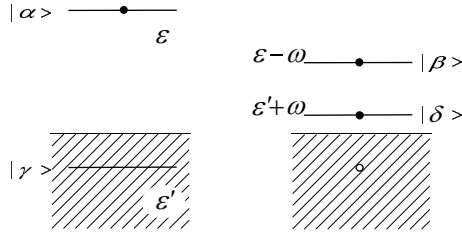


Fig. 15. A quasiparticle in a state $|\alpha\rangle$ of energy $\epsilon_\alpha = \epsilon$ interacts with another quasiparticle $|\gamma\rangle$ of energy $\epsilon_\gamma = \epsilon'$ in the Fermi sea. The final state is made of two quasiparticles above the Fermi sea and one hole.

$$\frac{1}{\tau_\alpha} = 2\pi s \sum_{\beta\gamma\delta} |\langle\alpha\gamma|U|\beta\delta\rangle|^2 \delta(\epsilon_\alpha + \epsilon_\gamma - \epsilon_\beta - \epsilon_\delta) \quad , \quad (93)$$

with the constraint that $\epsilon_\gamma < 0$, $\epsilon_\beta > 0$ and $\epsilon_\delta > 0$. If the matrix element does not depend on energies, we see immediately that the Landau ϵ^2 dependence comes simply from energy constraints : basically three final states can be chosen in a range of energy ϵ , with the constraint of energy conservation, whence the ϵ^2 dependence. Indeed the matrix element of the interaction is

energy independent in the ballistic case, but this is no true anymore in the diffusive case. The goal of the following pages is to calculate the energy dependence of the lifetime in the diffusive regime. Since we do not specify a given state $|\alpha\rangle$, eq. (93) can be rewritten for the lifetime at a given energy ϵ

$$\frac{1}{\tau_{ee}(\epsilon)} = \frac{1}{\nu_0} \sum_{\alpha} \frac{1}{\tau_{\alpha}} \delta(\epsilon - \epsilon_{\alpha}) = \frac{2\pi s}{\nu_0} \sum_{\alpha\beta\gamma\delta} |\langle\alpha\gamma|U|\beta\delta\rangle|^2 \delta(\epsilon_{\alpha} + \epsilon_{\gamma} - \epsilon_{\beta} - \epsilon_{\delta}) \delta(\epsilon - \epsilon_{\alpha}) , \quad (94)$$

where $\nu_0 = \rho_0 \Omega$ is the density of states. By introducing two energy integrals, it can be rewritten in the form

$$\frac{1}{\tau(\epsilon)} = \frac{2\pi s}{\nu_0} \int_0^{\epsilon} d\omega \int_{-\omega}^0 d\epsilon' \sum_{\alpha\beta\gamma\delta} |\langle\alpha\gamma|U|\beta\delta\rangle|^2 \delta(\epsilon - \epsilon_{\alpha}) \delta(\epsilon' - \epsilon_{\gamma}) \delta(\epsilon - \omega - \epsilon_{\beta}) \delta(\epsilon' + \omega - \epsilon_{\delta}) , \quad (95)$$

where $\epsilon > 0$, $\epsilon' < 0$ and $\epsilon - \omega > 0$, $\epsilon' + \omega > 0$ are respectively the two energies of the initial states and of the final states. If the matrix element is energy independent, we trivially recover the ϵ^2 dependence. Upon averaging over disorder, the lifetime has the form

$$\frac{1}{\tau_{ee}(\epsilon)} = 4\pi\nu_0^3 \int_0^{\epsilon} \omega W^2(\omega) d\omega \quad (96)$$

with

$$W^2(\omega) = \frac{1}{\nu_0^4} \overline{\sum_{\alpha\beta\gamma\delta} |\langle\alpha\gamma|U|\beta\delta\rangle|^2 \delta(\epsilon - \epsilon_{\alpha}) \delta(\epsilon' - \epsilon_{\gamma}) \delta(\epsilon - \omega - \epsilon_{\beta}) \delta(\epsilon' + \omega - \epsilon_{\delta})} \quad (97)$$

The characteristic matrix element $W(\omega)$ depends only on energy transfer ω , but neither on ϵ , nor on ϵ' .

The matrix element $\langle\alpha\gamma|U|\beta\delta\rangle$ can be evaluated on the basis of eigenfunction of the non interacting particles and reads

$$\langle\alpha\gamma|U|\beta\delta\rangle = \int d\mathbf{r}_1 d\mathbf{r}_2 \phi_{\alpha}^*(\mathbf{r}_1) \phi_{\gamma}^*(\mathbf{r}_2) \phi_{\beta}(\mathbf{r}_1) \phi_{\delta}(\mathbf{r}_2) U_{\omega}(\mathbf{r}_1 - \mathbf{r}_2) , \quad (98)$$

where $U_{\omega}(\mathbf{r})$ is the dynamically screened potential. The combination of wave functions and δ function can be rewritten in terms of Green's functions

$$-\frac{1}{\pi} \text{Im} G(\mathbf{r}, \mathbf{r}') = \sum_{\alpha} \phi_{\alpha}^*(\mathbf{r}) \phi_{\alpha}(\mathbf{r}') \delta(\epsilon - \epsilon_{\alpha}) , \quad (99)$$

so that $W^2(\omega)$ can be rewritten in the form

$$W^2(\omega) = \frac{1}{\nu_0^4 \pi^4} \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}'_1 d\mathbf{r}'_2 U_{\omega}(\mathbf{r}_1 - \mathbf{r}_2) U_{\omega}(\mathbf{r}'_1 - \mathbf{r}'_2) \\ \times \overline{\text{Im} G_{\epsilon}(\mathbf{r}_1, \mathbf{r}'_1) \text{Im} G_{\epsilon - \omega}(\mathbf{r}'_1, \mathbf{r}_1)} \overline{\text{Im} G_{\epsilon'}(\mathbf{r}_2, \mathbf{r}'_2) \text{Im} G_{\epsilon' + \omega}(\mathbf{r}'_2, \mathbf{r}_2)} \quad (100)$$

where the average the the product of four Green's functions has been decoupled into the product of two average values. By pairing Green's functions, it is possible to show that the average product contains a long-range part related to the probability P (see eq. 14 and [1])

$$\overline{\text{Im}G_\epsilon(\mathbf{r}, \mathbf{r}')\text{Im}G_{\epsilon-\omega}(\mathbf{r}', \mathbf{r})} = \pi\rho_0 \text{Re}P(\mathbf{r}, \mathbf{r}', \omega) . \quad (101)$$

We deduce

$$\begin{aligned} W^2(\omega) &= \frac{1}{\pi^2\nu_0^2\Omega^2} \int d\mathbf{r}_1 d\mathbf{r}_2 d\mathbf{r}'_1 d\mathbf{r}'_2 U_\omega(\mathbf{r}_1 - \mathbf{r}_2) U_\omega(\mathbf{r}'_1 - \mathbf{r}'_2) \\ &\quad \times \text{Re}P_d(\mathbf{r}_1, \mathbf{r}'_1, \omega) \text{Re}P_d(\mathbf{r}_2, \mathbf{r}'_2, -\omega) \end{aligned} \quad (102)$$

or, upon Fourier transformation :

$$W^2(\omega) = \frac{1}{\pi^2\nu_0^2\Omega^2} \sum_{\mathbf{q} \neq 0} |U(\mathbf{q}, \omega)|^2 [\text{Re}P_d(\mathbf{q}, \omega)]^2 . \quad (103)$$

At the diffusion approximation, the dynamically screened potential is given by (80) so that

$$W^2(\omega) = \frac{1}{4\pi^2\nu_0^4} \sum_{\mathbf{q} \neq 0} \frac{1}{\omega^2 + D^2q^4} , \quad (104)$$

which can be expressed in terms of the return probability $P(t)$

$$W^2(\omega) = \frac{1}{4\pi^2\nu_0^4} \frac{1}{\omega} \int_0^\infty P(t) \sin \omega t \, dt . \quad (105)$$

Finally, the electronic lifetime (96) is given by

$$\frac{1}{\tau_{ee}(\epsilon)} = \frac{2}{\pi\nu_0} \int_0^\infty \frac{P(t)}{t} \sin^2 \frac{\epsilon t}{2} dt \quad (106)$$

For a metal of volume Ω , we can identify two different regimes :

- $\epsilon \gg E_c$ where E_c is the Thouless energy. This corresponds to time scales $t \ll \tau_D$. In this case an electron described as a diffusive wave packet is insensitive to the boundaries and behaves as in an infinite medium where, according to (20), $P(t) = \Omega/(4\pi Dt)^{d/2}$. We obtain for the integral (105) : [24]

$$W^2(\omega) = \frac{dc_d}{16} \frac{1}{\nu_0^4 \omega^2} \left(\frac{\omega}{E_c} \right)^{d/2} , \quad (107)$$

so that the electronic lifetime is equal to

$$\frac{1}{\tau_{ee}(\epsilon)} = \frac{\pi}{2} c_d \Delta \left(\frac{\epsilon}{E_c} \right)^{d/2} \quad (\epsilon \gg E_c) \quad (108)$$

where $c_1 = \sqrt{2}/\pi^2$, $c_2 = 1/4\pi^2$, $c_3 = \sqrt{2}/6\pi^3$. Such a behavior has been indeed observed in silver wires ($d = 1$) for which $W^2(\omega) \propto \omega^{-3/2}$ and $1/\tau_{ee}(\epsilon) \propto \epsilon^{1/2}$, although the measured prefactor came out to be larger than the value predicted here [25].

• The limit $\epsilon \ll E_c$, that is $t \gg \tau_D$, corresponds to the ergodic regime in which the diffusive electronic wave packet explores all the accessible volume Ω . Thus we would expect $P(t)$ to be driven only by the zero mode. This is not so, because in expression (104) this mode has been removed in order to ensure electronic neutrality. The excitation energy ϵ is smaller than E_c and it is not possible to replace the sum (104) by an integral. In this limit, we obtain

$$W^2(\omega) = \frac{a_d}{4\pi^6} \frac{\Delta^4}{E_c^2} \propto \frac{\Delta^2}{g^2} , \quad (109)$$

where the coefficient a_d is defined by the series

$$a_d = \sum_{n_x, n_y, n_z} \frac{1}{(n_x^2 + n_y^2 + n_z^2)^2} . \quad (110)$$

The ratio E_c/Δ is the dimensionless conductance g . For $\omega \ll E_c$, the characteristic matrix element of the interaction is thus energy independent and of order Δ/g . The inverse lifetime in this case is [26]

$$\frac{1}{\tau_{ee}(\epsilon)} = \frac{a_d}{2\pi^5} \Delta \left(\frac{\epsilon}{E_c} \right)^2 \quad (\epsilon \ll E_c) . \quad (111)$$

9.5 Quasiparticle lifetime at finite temperature

In Landau theory it is well-known that the quasiparticle lifetime at zero energy $\epsilon = 0$ and finite temperature T is simply obtained by replacing ϵ by T , so that it varies as T^2 . The diffusive case is more subtle. It turns out that in this case we cannot simply substitute ϵ by T . This is wrong in low dimension. Let us see why.

What is changed at finite temperature? The Fermi golden rule is modified to account for Fermi factors. The condition of filled or empty states has to be replaced by Fermi factors and eq. (95) generalizes as

$$\frac{1}{\tau_{ee}(\epsilon, T)} = 4\pi\nu_0^3 \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\epsilon' F(\epsilon, \epsilon', \omega) W^2(\omega) \quad (112)$$

where $F(\epsilon, \epsilon', \omega)$ is a combination of Fermi factors $f_\epsilon = 1/(e^{\beta\epsilon} + 1)$:

$$F(\epsilon, \epsilon', \omega) = f_{\epsilon'}(1 - f_{\epsilon-\omega})(1 - f_{\epsilon'+\omega}) + (1 - f_{\epsilon'})f_{\epsilon-\omega}f_{\epsilon'+\omega} . \quad (113)$$

The first term in this expression is larger when $\epsilon > 0$. It describes the decay of an electron-like state above the Fermi level. The second term dominates when $\epsilon < 0$ and describes the decay of a hole-like state into the Fermi sea. For $\epsilon = 0$, both terms are equal. Integrating upon ϵ' , we obtain

$$\frac{1}{\tau_{ee}(\epsilon, T)} = 4\pi\nu_0^3 \int_{-\infty}^{\infty} d\omega \, \omega W^2(\omega) f_{\epsilon-\omega} \frac{e^{\beta\epsilon} + 1}{e^{\beta\omega} - 1} . \quad (114)$$

This lifetime can also be obtained from the imaginary part of the self-energy of a quasiparticle in the presence of a screened interaction [27]. At zero temperature, we recover the result (108).

9.6 Quasiparticle lifetime at the Fermi level

We now consider more specifically the lifetime of a quasiparticle *at the Fermi level* ($\epsilon = 0$) *and at finite temperature*. Physical properties such as conductance are expressed in terms of *single particle* states at the Fermi level. It is thus essential to understand the range of validity of the description in terms of independent quasiparticles. From relation (114), we have [28]

$$\frac{1}{\tau_{ee}(T)} = 8\pi\nu_0^3 \int_0^{\infty} d\omega W^2(\omega) \frac{\omega}{\sinh \beta\omega} . \quad (115)$$

For the diffusion in free space, the matrix element $W^2(\omega)$ is proportional to $\omega^{d/2}$ (relation 107), so that

$$\frac{1}{\tau_{ee}(T)} = \frac{\pi d c_d}{2\nu_0} \int_0^{\infty} \frac{d\omega}{\omega \sinh \beta\omega} \left(\frac{\omega}{E_c} \right)^{d/2} . \quad (116)$$

Therefore, in three dimensions, we have

$$\frac{1}{\tau_{ee}(T)} = \frac{\sqrt{2}}{4\pi^2\nu_0} \int_0^{\infty} \frac{d\omega}{\omega \sinh \beta\omega} \left(\frac{\omega}{E_c} \right)^{3/2} \simeq \frac{T}{\nu_0} \int_0^T \frac{d\omega}{\omega^2} \left(\frac{\omega}{E_c} \right)^{3/2} , \quad (117)$$

that is

$$\frac{1}{\tau_{ee}(T)} \simeq \Delta \left(\frac{T}{E_c} \right)^{3/2} \quad (d = 3) \quad (118)$$

up to a numerical factor. Note that the exponent of the power law is the same as the exponent for the energy dependence of the lifetime at zero temperature (108). This result follows at once if we notice that relevant processes in the quasiparticle relaxation described by $\omega W^2(\omega)$ are those for which the energy transfer ω is of order T .

It would be tempting to generalize this result to any dimension and to conclude that $1/\tau_{ee}(T) \propto T^{d/2}$. This is not correct for $d \leq 2$. In this case, the contribution of e-e processes with *low energy transfer* $\omega \simeq 0$ dominates and

leads to a divergence in the integral (116). In order to cure this divergence, it is worth noticing that $\tau_{ee}(T)$ represents precisely the lifetime of an eigenstate, so that the energy transfer ω cannot be defined with an accuracy better than $1/\tau_{ee}$. Consequently, there is no energy transfer smaller than $1/\tau_{ee}(T)$, so that the integral (116) needs to be cut off self-consistently for ω smaller than $1/\tau_{ee}(T)$. For $d \leq 2$, we thus obtain a self-consistent relation for τ_{ee} :

$$\frac{1}{\tau_{ee}(T)} \simeq \frac{1}{\nu_0} \int_{1/\tau_{ee}}^{\infty} \frac{d\omega}{\omega \sinh \beta \omega} \left(\frac{\omega}{E_c} \right)^{d/2} \simeq \frac{T}{\nu_0} \int_{1/\tau_{ee}}^T \frac{d\omega}{\omega^2} \left(\frac{\omega}{E_c} \right)^{d/2} \quad (119)$$

where the thermal factor has been replaced by a cutoff at $\omega \sim T$. In two dimensions, $1/\tau_{ee}(T)$ is proportional to the temperature (within logarithmic corrections) :

$$\frac{1}{\tau_{ee}(T)} \simeq \Delta \frac{T}{E_c} \ln \frac{E_c}{\Delta} \quad (d = 2) \quad . \quad (120)$$

In one dimension, and since $T\tau_{ee} \gg 1$, the integral becomes proportional to $\sqrt{\tau_{ee}}$ so that the self-consistent relation leads to

$$\frac{1}{\tau_{ee}(T)} \simeq \Delta \left(\frac{E_c}{\Delta} \right)^{1/3} \left(\frac{T}{E_c} \right)^{2/3} \quad (d = 1) \quad . \quad (121)$$

9.7 Phase coherence

The time (115) has been defined as the lifetime of a quasiparticle, generalizing the notion introduced by Landau to the case of a diffusive system in d dimensions. We should now evaluate the phase coherence time $\tau_\phi(T)$ which limits coherent effects like the weak localization correction (34). This time can be interpreted as the lifetime of the Cooperon. Its derivation consists in calculating *directly* the dephasing $\langle e^{i\Phi(t)} \rangle$ resulting from electron-electron interaction and accumulated between time reversed conjugated multiple scattering sequences. To that purpose, the interaction between electrons is replaced by an effective interaction which describes the coupling of a single electron to the electromagnetic field created by the other electrons. This calculation [12, 29] is not developed here; see [1] for a detailed derivation.

An alternative and qualitative approach is to consider that phase coherence is limited by the lifetime of quasiparticles. Since the multiple scattering trajectories that are paired in the Cooperon are defined for a given energy state, they cannot interfere for times larger than $\tau_{ee}(T)$. This results in an irreversible dephasing between the trajectories and thus a loss of phase coherence. It is therefore natural to assume that

$$\tau_\phi(T) = \tau_{ee}(T) \quad . \quad (122)$$

Indeed, the temperature dependences predicted in equations (118,120,121) have been confirmed experimentally, in all dimensions by weak localization measurements.

It turns out that not only these two characteristic times $\tau_{ee}(T)$ and $\tau_\phi(T)$ are equal (within a numerical factor), but also that the two processes, *quasi-particles relaxation* and *phase relaxation*, are very similar. Finally let us remark that the introduction of the low-energy cutoff in (119) may appear as a handwaving and artificial way to handle the low energy divergence. We have shown recently that the profound reason for this divergence is that, for $d \leq 2$, *relaxation of quasiparticles* as well as the *phase relaxation* are not exponential [30].

Acknowledgments - Many points of view presented in these lectures have been developed during a long collaboration with Eric Akkermans, and are detailed in ref. [1].

References

1. For a comprehensive review, see E. Akkermans and G. Montambaux, *Mesoscopic physics of electrons and photons*, Cambridge University Press (2007)
2. Y. Imry and R.A. Webb, *Quantum interference and the Aharonov-Bohm effect*, Scientific American, **260**, 36 (1989); M. Peshkin and A. Tonomura, *The Aharonov-Bohm effect*, Lecture notes in Physics **340**, Springer-Verlag, Heidelberg, 1989
3. R.A. Webb, S. Washburn, C.P. Umbach and R.P. Laibowitz, *Observation of h/e Aharonov-Bohm oscillations in normal-metal rings*, Phys. Rev. Lett. **54**, 2696 (1985)
4. D.Yu. Sharvin and Yu.V. Sharvin, *Magnetic-flux quantization in a cylindrical film of a normal metal*, JETP Lett. **34**, 272 (1981)
5. Y. Aharonov and D. Bohm, *Significance of electromagnetic potentials in quantum theory*, Phys. Rev. **115**, 485 (1959)
6. B. Pannetier, J. Chaussy, R. Rammal and P. Gandit, *Magnetic flux quantization in the weaklocalization regime of a nonsuperconducting metal*, Phys. Rev. Lett. **53**, 718 (1984); B. Douçot and R. Rammal, *Quantum Oscillations in Normal-Metal Networks*, Phys. Rev. Lett., **55**, 1148 (1985);
7. M. Büttiker, *Symmetry of electrical conduction*, IBM J. Res. Develop. **32**, 317 (1988)
8. C.W.J. Beenakker, *Random-matrix theory of quantum transport*, Rev. Mod. Phys. **69**, 731 (1997)
9. R.P. Feynman and A.R. Hibbs, *Quantum mechanics and path integrals*, McGraw-Hill, New-York (1965); G. Roepstorff, *Path integral approach to quantum physics : an introduction*, Springer Verlag, New-York (1994)
10. D.E. Khmelnitskii, *Localization and coherent scattering of electrons*, Physica **126 B**, 235 (1984)
11. G. Bergmann, *Weak-localisation in thin films*, Phys. Rep. **107**, 1 (1984)
12. S. Chakravarty and A. Schmid, *Weak-localization: the quasiclassical theory of electrons in a random potential*, Phys. Rep. **140**, 193 (1986)

13. Notice that this correct result differs from the qualitative evaluation (33) only by a factor 2.
14. B. Kramer and A. MacKinnon, *Localization : theory and experiment*, Rep. Prog. Phys. **56**, 1469 (1993)
15. M. Abramowitz and I.A. Stegun, *Handbook of mathematical functions*, Dover (1972).
16. B.L. Altshuler, A.G. Aronov and B.Z. Spivak, *The Aharonov-Bohm effect in disordered conductors*, JETP Lett. **33**, 94 (1981)
17. D. Mailly and M. Sanquer, *Sensitivity of quantum conductance fluctuations and $1/f$ noise to time reversal symmetry*, J. Physique I France **2**, 357 (1992)
18. M. Pascaud and G. Montambaux, *Persistent currents on networks*, Phys. Rev. Lett. **82**, 4512 (1999); E. Akkermans, A. Comtet, J. Desbois, G. Montambaux and C. Texier, *Spectral determinant on quantum graphs*, Ann. Phys. **284**, 10 (2000)
19. G. Montambaux and A. Akkermans, *Quantum transport in disordered conductors*, "Nanotubes : Science and Applications ", Eds. A. Loiseau, P. Launois, P. Petit, S. Roche, J.P. Salvetat (Springer Verlag-Frontiers in Physics, 2006)
20. B. L. Altshuler and A. G. Aronov, *Electron-electron interaction in disordered conductors*, in "Electron-electron interactions in disordered systems", A.L. Efros and M. Pollak eds., p. 1 (Elsevier 1985)
21. There is also a contribution in the Cooperon channel. However, this contribution is corrected by higher order diagrams and turns out to be small.
22. Y. Imry and Z. Ovadyahu, *Density of states anomalies in a disordered conductor : a tunneling study*, Phys. Rev. Lett. **49**, 841 (1982); F. Pierre, H. Pothier, P. Joyez, N.O. Birge, D. Esteve and M. Devoret, *Electrodynamic dip in the local density of states of a metallic wire*, Phys. Rev. Lett. **86**, 1590 (2001)
23. From our simple picture, the coefficients λ_σ and λ_ρ are equal. A better description shows that these parameters are slightly different, see [20, 1].
24. Alternatively, the sum (104) on wavevectors can be replaced by an integral and we recover (107).
25. F. Pierre, H. Pothier, D. Esteve and M.H. Devoret, *Energy redistribution between quasiparticles in mesoscopic silver wires*, J. Low Temp. Phys., **118**, 437 (2000); F. Pierre, *Interactions électron-électron dans les fils mésoscopiques*, Ann. Phys. **26**, vol.4 (2001)
26. U. Sivan, Y. Imry and A.G. Aronov, *Quasi-particle lifetime in a quantum dot*, Europhys. Lett. **28**, 115 (1994)
27. E. Abrahams, P.W. Anderson, P.A. Lee and T.V. Ramakrishnan, *Quasiparticle lifetime in disordered two-dimensional metals*, Phys. Rev. B **24**, 6783 (1981)
28. W. Eiler, *Electron-electron interaction and weak localization*, J. Low Temp. Phys. **56**, 481 (1984)
29. B.L. Altshuler, A.G. Aronov and D.E. Khmelnitskii, *Effects of electron-electron collisions with small energy transfers on quantum localisation*, J. Phys. C **15**, 7367 (1982)
30. G. Montambaux and E. Akkermans, *Non exponential quasiparticle decay and phase relaxation in low dimensional conductors*, Phys. Rev. Lett. **95**, 016403 (2005); C. Texier and G. Montambaux, *Theory of quantum transport in multi-terminal networks of diffusive wires*, in preparation.